Detection and Estimation of Generalized Chirps Using Time-Frequency Representations*

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ABSTRACT

We propose techniques for the detection and parameter estimation of generalized chirps in the presence of noise. Generalized chirps are nonstationary signals characterized by group delays with specific dispersion law characteristics. Special cases of generalized chirps include linear chirps, and hyperbolic chirps that are Doppler-invariant signals. We optimally detect generalized chirps using generalized time-shift covariant quadratic time-frequency representations (QTFRs) such as hyperbolic QTFRs used for detecting hyperbolic chirps. We also propose the parameter estimation of generalized chirps, and specialize our simulation results to hyperbolic chirps. We combine phase unwrapping with linear regression of the phase data at high signal-to-noise ratios (SNRs) to produce very simple and unbiased estimators that attain the Cramer-Rao lower bounds on variance. Maximum likelihood estimation performs well at low SNRs, but at the cost of high computational complexity.

1 INTRODUCTION

Nonstationary signals have been analyzed successfully using quadratic time-frequency representations (QTFRs) [1]. QTFR theory can also be used, however, for detecting nonstationary signals [2]-[5], and for estimating their unknown parameters in the presence of noise [2], [4]-[9]. The optimal detection of linear chirps using QTFRs was derived independently by Kay and Boudreaux-Bartels in [2] and Kumar and Carroll in [3] using the Wigner distribution. Also, a time-frequency (TF) formulation of optimal receivers was proposed by Flandrin in [4] using Cohen's class of TF shift covariant QTFRs and by Papandreou, Kay and Boudreaux-Bartels in [5] using the hyperbolic class of scale and hyperbolic time-shift covariant QTFRs. Carrying out the detection in the TF plane often results in simple and intuitive receiver structures and permits optimal detection for a broader class of signals than classical detectors. The estimation of unknown parameters of a complex sinusoid, linear chirp, and polynomial phase chirp signal using phase unwrapping plus linear regression of the phase data at high signal-to-noise ratios (SNRs) was achieved by Tretter [6] and Kay [7], Djurić and Kay [8], and Boashash [9], respectively.

Generalized chirps are important nonstationary signals as they reflect a specific dispersion law characterized by the shape of their group delay [10]. They are defined in the frequency domain as

$$G_c(f) \triangleq \sqrt{|\tau(f)|}\, e^{j2\pi c\,\xi(\frac{f}{f_r})} \qquad (1)$$

where $c \in \mathbb{R}$ is their generalized parameter, and their group delay $c\,\tau(f)=\frac{c}{f_r}\xi'(\frac{f}{f_r})$ is proportional to $\xi'(b)=\frac{d}{db}\xi(b)$, the derivative of the invertible phase function $\xi(b)$. Here, $f_r > 0$ is a reference frequency. A desirable property of the generalized chirp is that applying a generalized time-shift to it simply changes its parameter c, i.e., if $(\mathcal{G}_{c_0}G_c)(f)=e^{-j2\pi c_0\xi(f/f_r)}G_c(f)$ then $(\mathcal{G}_{c_0}G_c)(f)=G_{c-c_0}(f)$.

Generalized time-shift covariant QTFRs (or simply generalized class QTFRs) are ideal for analyzing generalized chirps. Any generalized class QTFR, $T_x^{(G)}(t,f)$, of a signal $x(t)$ (with Fourier transform $X(f)$) satisfies[1] the generalized time-shift covariance property $T_{\tilde{x}}^{(G)}(t,f)=T_x^{(G)}(t - c\tau(f),f)$ where $\tilde{X}(f)=(\mathcal{G}_c X)(f)$, and it is defined as [10]

$$T_x^{(G)}(t,f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_x^{(G)}(t_1,f_1) \cdot\psi_T^{(G)}\Big(\frac{t}{\tau(f)} - \frac{t_1}{\tau(f_1)}, \xi(\frac{f}{f_r}) - \xi(\frac{f_1}{f_r})\Big) dt_1\, df_1 \qquad (2)$$

where $\psi_T^{(G)}(c,b)$ is a 2-D kernel characterizing the QTFR $T^{(G)}$. The generalized time-shift version of the Wigner distribution (GCWD) is given by

$$Q_x^{(G)}(t,f) = \int_{-\infty}^{\infty} X(f_r d(f,\beta))X^*(f_r d(f,-\beta)) \cdot\frac{f_r e^{j2\pi\frac{t}{\tau(f)}\beta}}{\sqrt{|\xi'(d(f,\beta))\xi'(d(f,-\beta))|}}\, d\beta \qquad (3)$$

[1]The generalized time-shift covariant QTFRs considered here also satisfy a covariance to warped frequency-shifts [11, 12, 10] which correspond to TF scalings for the special case of the hyperbolic class where $\tau(f)=1/f$.

*Funding by ONR grant N00014-92-J-1499, and by AFOSR 93-1-0006.

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Chirp	Phase function, $\xi(b)$ in (1)	Matched QTFR, $Q_{g_c}^{(G)}(t,f)$ in (3)
Generalized	arbitrary	$\|\tau(f)\|\,\delta(t+c\tau(f))$, GCWD
Hyperbolic	$\ln b$	$\frac{1}{f}\,\delta(t+\frac{c}{f})$, $f>0$, Altes distribution
Complex sinusoid	b	$\frac{1}{f_r}\,\delta(t+\frac{c}{f_r})$, Wigner distribution (WD)
Linear FM	b^2	$\|\frac{2f}{f_r^2}\|\,\delta(t+\frac{2cf}{f_r^2})$, $f>0$
κth Power, $\kappa>0$	b^κ	$\|\frac{\kappa}{f_r}(\frac{f}{f_r})^{\kappa-1}\|\,\delta(t+\frac{\kappa c}{f_r}(\frac{f}{f_r})^{\kappa-1})$, $f>0$, Power WD

Table 1: Generalized chirps for different phase functions and their characteristic matched QTFR. Here, $\tau(f)$ is the group delay function, which is proportional to $\xi'(f/f_r)$.

with $d(f,\beta)=\xi^{-1}(\xi(f/f_r)+\beta/2)$ where $\xi^{-1}(\xi(b))=b$. The GCWD of a generalized chirp $g_c(t)$, whose Fourier transform is given in (1), is a 2-D delta function centered along the group delay curve $t=-c\tau(f)$,

$$Q_{g_c}^{(G)}(t,f) = |\tau(f)|\;\delta(t+c\,\tau(f)). \quad (4)$$

For example, the generalized chirps characterized by the hyperbolic group delay $\tau(f)=1/f$ and phase function $\xi(b)=\ln b$ in (1) are the hyperbolic chirps

$$H_c(f) = \frac{1}{\sqrt{f}}\,e^{j2\pi c\,\ln\frac{f}{f_r}}, \quad f>0. \quad (5)$$

They are Doppler-invariant signals since scaling them only changes their phase, i.e., $(\mathcal{C}_a H_c)(f) = 1/\sqrt{|a|}\,H_c(f/a) = e^{-j2\pi c\ln a}H_c(f)$. Applying a hyperbolic time-shift simply shifts their parameter, i.e., $(\mathcal{H}_{c_0}H_c)(f)=e^{-j2\pi c_0\ln\frac{f}{f_r}}H_c(f) = H_{c-c_0}(f)$. The corresponding general class QTFRs that can be used for the analysis of hyperbolic chirps are the hyperbolic class QTFRs given in (2) with $\tau(f)=1/f$, $\xi(b)=\ln b$

$$T_x^{(H)}(t,f)=\int_0^{\infty}\int_{-\infty}^{\infty}Q_x^{(H)}(t_1,f_1)\psi_T^{(H)}\Big(tf-t_1f_1,\ln\frac{f}{f_1}\Big)\,dt_1\,df_1$$

for $f>0$ [11]. The hyperbolic class contains QTFRs that maintain hyperbolic time-shifts, and TF scalings of the analysis signal, respectively,

$$T_{\tilde{x}}^{(H)}(t,f)=T_x^{(H)}(t-c/f,f) \text{ with } \tilde{X}(f)=(\mathcal{H}_cX)(f),$$
$$T_{\tilde{x}}^{(H)}(t,f)=T_x^{(H)}(at,f/a) \text{ with } \tilde{X}(f)=(\mathcal{C}_aX)(f).$$

As a result, they are important for the analysis of hyperbolic chirps. The corresponding GCWD in the hyperbolic class is the Altes distribution (AD),

$$Q_x^{(H)}(t,f)=f\int_{-\infty}^{\infty}X(fe^{\beta/2})X^*(fe^{-\beta/2})e^{j2\pi tf\beta}d\beta,$$

which is the GCWD in (3) with $\xi(b)=\ln b$. The AD of a hyperbolic chirp $h_c(t)$, whose Fourier transform is given in (5), is a delta function centered along the hyperbolic group delay $t=-c/f$ (cf. (4)),

$$Q_{h_c}^{(H)}(t,f) = \frac{1}{f}\,\delta(t+\frac{c}{f}),\ f>0. \quad (6)$$

Other generalized chirps in (1) are complex sinusoids characterized by $\xi(b)=b$, and κth power chirps characterized by $\xi(b)=b^\kappa$ (e.g. linear chirps for $\kappa=2$) [10]. These are summarized together with their corresponding phase functions and analysis QTFRs in Table 1.

In this paper, we propose the use of generalized time-shift covariant QTFRs for the optimum detection of generalized chirps in the presence of noise, which simplifies to the use of hyperbolic class QTFRs for the special case of hyperbolic chirps. We also propose methods for the parameter estimation of generalized chirps, and specialize our simulation results to hyperbolic chirps. One method combines phase unwrapping with linear regression of the phase data at high SNRs to produce very simple and unbiased estimators that attain the Cramer-Rao lower bounds (CRLBs) on variance. Maximum likelihood estimation techniques perform well at low SNRs, but at the cost of high computational complexity.

2 OPTIMUM DETECTION

We want to detect generalized chirps (1) in the presence of noise. Thus, we formulate the detection problem of a complex nonstationary Gaussian Rayleigh fading signal $b\,g_c(t)$ in complex zero-mean white Gaussian noise $e(t)$ in terms of hypothesis testing

$$\left.\begin{array}{ll} H_0: & r(t)=e(t) \\ H_1: & r(t)=b\,g_c(t)+e(t) \end{array}\right\}\quad t\in(T) \quad (7)$$

where $r(t)$ is the observation signal on the time interval (T), and b is a zero-mean complex Gaussian random variable with independent real and imaginary parts of equal variances. The optimal solution of (7) (see [13]) states that H_1 is true if

$$l = \left|\int_{(T)} r(t)\;g_c^*(t)\,dt\right|^2 > \gamma \quad (8)$$

where the threshold γ is obtained from the probability of false-alarm using a Neyman-Pearson test [13]. The statistic in (8) is the inner product of the transmitted signal $r(t)$ and the generalized chirp $g_c(t)$. Thus, we can reformulate the optimum solution in (8) in terms of generalized class QTFRs that preserve inner products (Moyal's formula)

$$\left|\int_{-\infty}^{\infty} x_1(t) x_2^*(t) dt\right|^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_{x_1}^{(G)}(t,f) T_{x_2}^{(G)*}(t,f) dt df. \quad (9)$$

Generalized class QTFRs that satisfy (9) have a characteristic kernel $\psi_T^{(G)}(c,b)$ in (2) which satisfies $|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_T^{(G)}(c,b)\, e^{-j2\pi(\beta c - \zeta b)} db\, dc| = 1$. The kernel of the GCWD, $Q_x^{(G)}(t,f)$, is $\psi_Q^{(G)}(c,b) = \delta(b)\delta(c)$, and thus the GCWD (e.g. the AD) satisfies the Moyal's formula condition and thus (9). We can now reformulate the optimal solution (8) in terms of GCWDs

$$l = \int_{-\infty}^{\infty} \int_{(T)} Q_r^{(G)}(t,f)\, Q_{g_c}^{(G)}(t,f)\, dt\, df > \gamma. \quad (10)$$

The importance of this reformulation lies in the fact that for a generalized chirp, the GCWD in (4) is a delta function centered along its group delay simplifying the statistic in (10) for $t = -c\,\tau(f) \in (T)$ to

$$l = \int_{-\infty}^{\infty} Q_r^{(G)}(-c\tau(f), f)\, |\tau(f)|\, df > \gamma. \quad (11)$$

As a result, the statistic in (8) reduces to a very simple and intuitive form in the TF plane that involves a simple integration of the GCWD of the observation signal $r(t)$ along the transmitted signal's group delay $\tau(f)$. For the hyperbolic chirp in (5) and its AD in (6), (11) simplifies further to the integration of the observation signal's AD along a hyperbola. Note that this is an extension of the optimal linear chirp detection in [2] which involves integrating the Wigner distribution of the observed signal along the chirp signal's linear instantaneous frequency. As a result, the new formulation in the TF plane in (11) is advantageous in that it results in simple test statistics that can be computed in the TF plane where the nonstationary signal is analyzed.

3 PARAMETER ESTIMATION

The generalized parameter c of the generalized chirp in (1) is not always known. Here, we propose three methods for the parameter estimation of the generalized chirp parameters in the presence of noise that include phase unwrapping plus linear regression at high SNRs [6, 7, 8], double difference phase estimation [8], and maximum likelihood estimation techniques [14]. These methods have been applied previously to complex sinusoids and linear chirps, and are here extended for any generalized chirp. We compare them via simulations for the special case of hyperbolic chirps.

3.1 High SNR Linear Regression Estimator

In discrete frequency form, a generalized chirp in complex white Gaussian noise $E[k]$ (of variance σ_e^2) is

$$X[k] = \sqrt{|\tau[k]|}\; e^{j(2\pi c\,\xi[k] + 2\pi\eta k)} + E[k],$$

where $k = k_0, k_0+1, \ldots, N+k_0-1$ assuming a normalized initial time $|\eta| < 0.5$, length N, and initial frequency sample k_0. The problem is to estimate the unknown parameter vector $\underline{\boldsymbol{\theta}} = [c\ \eta]^{\mathrm{T}}$ (where T denotes transpose) given $X[k]$ and k_0. We obtain a simple estimator for the unknown parameters, similar to the one used in [8] for linear chirps, by using a high SNR approximation

$$X[k] \approx \sqrt{|\tau[k]|}\; e^{j(2\pi c\xi[k] + 2\pi\eta k) + W[k])}, \quad (12)$$

where $W[k]$ is real, zero-mean Gaussian noise with variance $\sigma_e^2/(2\,|\tau[k]|)$, $k = k_0, \ldots, N+k_0-1$. The phase data of the approximated generalized chirp (12) is linear with respect to the unknown parameters

$$\Phi[k] = 2\pi\, c\,\xi[k] + 2\pi\,\eta\, k + W[k], \quad (13)$$

for $k = k_0, \ldots, N+k_0-1$. Thus, by unwrapping the phase of (12), we greatly simplify the parameter estimation to a linear model in (13). A possible phase unwrapping algorithm is [8]

$$\begin{aligned}
&\bullet \Phi[k_0] = \arg X[k_0],\\
&\bullet \Phi[k_0+1] = \arg(X[k_0+1]\; X^*[k_0]) + \Phi[k_0],\\
&\bullet \Delta^2\Phi[k] = \arg(X[k](X^*[k-1])^2 X[k-2]), \quad (14)\\
&\bullet \Phi[k] = \Delta^2\Phi[k] + 2\Phi[k-1] - \Phi[k-2],
\end{aligned}$$

where $\Delta^2\Phi[k] = \Phi[k] - 2\Phi[k-1] + \Phi[k-2]$ is the double difference phase, "arg" refers to the inverse tangent, and $k \epsilon (K) = \{k_0+2, \ldots, N+k_0-1\}$. Note that other phase unwrapping algorithms such as higher phase differences can be used to obtain the phase of (12), that are more suitable to the particular phase function $\xi[k]$. In [8], it was mentioned that the phase unwrapping scheme can be used for $\xi[k]$ polynomials of any degree [9]. Here, we also use it for hyperbolic phase functions. If the above scheme for phase unwrapping is used, then it is important to take into account the following three constraints on the allowable range of the unknown parameter c that arise since the inverse tangent function is used

$$|\,2\pi c\, \Delta^2 \xi[k]\,| < \pi, \quad k\,\epsilon\,(K) \quad (15)$$

$$|\,2\pi c\,\xi[k_0] + 2\pi\,\eta\, k_0\,| < \pi \quad (16)$$

$$|\,2\pi c\,(\xi[k_0+1] - \xi[k_0]) + 2\pi\,\eta\,| < \pi \quad (17)$$

where $\Delta^2\xi[k] = \xi[k] - 2\xi[k-1] + \xi[k-2]$. Since desirably $|c| < \infty$, the first constraint in (15) limits c depending on the k_0 value if $|\Delta^2\xi[k]|$ is a monotonically decreasing function for $k\,\epsilon\,(K)$. If $|\Delta^2\xi[k]|$ has a maximum at $k = \hat{k}$, then the limits on c depend on the $\hat{k}$ value. Alternative phase unwrapping algorithms are needed when $|\Delta^2\xi[k]|$ is monotically increasing. Also note that if $\Delta^2\xi[k]$ does not depend on k, the constraints in (16)-(17) are not necessary, and thus fewer limitations are imposed on the allowable range of c.

Having obtained $\Phi[k]$ in (14), the general linear model (13) in vector form is $\underline{\Phi} = \mathbf{H}\underline{\theta} + \underline{\mathbf{W}}$ where the observation matrix is

$$\mathbf{H} = 2\pi \begin{bmatrix} \xi[k_0] & \xi[k_0+1] & \dots & \xi[N+k_0-1] \\ k_0 & k_0+1 & \dots & N+k_0-1, \end{bmatrix}^{\mathrm{T}},$$

$\underline{\Phi} = [\Phi[k_0]\,\Phi[k_0+1] \ \dots \ \Phi[N+k_0-1]]^{\mathrm{T}}$, and $\underline{\mathbf{W}}$ is the noise vector with covariance matrix $\mathbf{C}_{\underline{\mathbf{W}}} = \frac{\sigma_e^2}{2}\,\mathrm{diag}\left(\frac{1}{|\tau[k_0]|}, \dots, \frac{1}{|\tau[N+k_0-1]|}\right)$. The "best" estimator at high SNR is given by [14]

$$\underline{\hat{\theta}} = [\hat{c}\ \hat{\eta}]^{\mathrm{T}} = (\mathbf{H}^{\mathrm{T}}\,\mathbf{C}_{\underline{\mathbf{W}}}^{-1}\,\mathbf{H})^{-1}\,\mathbf{H}^{\mathrm{T}}\,\mathbf{C}_{\underline{\mathbf{W}}}^{-1}\,\underline{\Phi}. \tag{18}$$

In addition to its simple to compute structure, the estimator in (18) at high SNR is unbiased and attains the CRLB [15]. We derived the CRLB on the variance, $\mathrm{Var}(\cdot)$, based on the Fisher information matrix [14] for this estimation problem to be

$$\mathrm{Var}(\hat{c}) \ \geq \ \frac{\sigma_e^2}{2}\frac{1}{4\pi^2}\frac{P}{PQ-S^2} \tag{19}$$

$$\mathrm{Var}(\hat{\eta}) \ \geq \ \frac{\sigma_e^2}{2}\frac{1}{4\pi^2}\frac{Q}{PQ-S^2} \tag{20}$$

where $P = \sum_{k=k_0}^{N+k_0-1} |\tau[k]|\,k^2$, $Q = \sum_{k=k_0}^{N+k_0-1} |\tau[k]|\,\xi^2[k]$, and $S = \sum_{k=k_0}^{N+k_0-1} k\,|\tau[k]|\,\xi[k]$. It can be shown that the inverse of the Fisher information matrix is identical to the covariance matrix of the estimates [15]. Thus, at high SNR, the variance of the estimates is given by (19) and (20) when the equality of the CRLB holds.

3.2 Double Difference Phase Estimator

If we only want to estimate the generalized parameter c, then we can avoid the extra step of phase unwrapping altogether, and thus eliminate the constraints (16)-(17) on the allowable range of c. We achieve the estimation using the double difference phase data $\Delta^2\Phi[k]$ computed from the data using (14), noting its equivalence to

$$\Delta^2\Phi[k] = 2\pi c\,\Delta^2\xi[k] + V[k],\ k \in (K)$$

where $V[k] = \Delta^2 W[k] = W[k] - 2W[k-1] + W[k-2]$. Thus, the only constraint on c is (15) which arises since $|\Delta^2\Phi[k]| < \pi$ due to the inverse tangent function. The resulting linear model estimator in vector form is $\underline{\Psi} = c\underline{\mathbf{u}} + \underline{\mathbf{V}}$ where $\underline{\Psi} = [\Delta^2\Phi[k_0+2]\,\Delta^2\Phi[k_0+3] \dots \Delta^2\Phi[N+k_0-1]]^{\mathrm{T}}$,

$$\underline{\mathbf{u}} = 2\pi\left[\Delta^2\xi[k_0+2]\ \Delta^2\xi[k_0+3] \dots \Delta^2\xi[N+k_0-1]\right]^{\mathrm{T}}.$$

and $\underline{\mathbf{V}} = [\Delta^2 W[k_0+2] \dots \Delta^2 W[N+k_0-1]]^{\mathrm{T}}$. The resulting estimator for c is given by

$$\hat{c} = (\underline{\mathbf{u}}^{\mathrm{T}}\,\mathbf{C}_{\underline{\mathbf{V}}}^{-1}\,\underline{\Psi})\ /\ (\underline{\mathbf{u}}^{\mathrm{T}}\,\mathbf{C}_{\underline{\mathbf{V}}}^{-1}\,\underline{\mathbf{u}}), \tag{21}$$

where $\mathbf{C}_{\underline{\mathbf{V}}}$ is the covariance matrix of the zero-mean Gaussian noise vector $\underline{\mathbf{V}}$ [15]. At high SNR, the estimator is unbiased and its variance is $\mathrm{Var}(\hat{c}) = 1/\underline{\mathbf{u}}^{\mathrm{T}}\,\mathbf{C}_{\underline{\mathbf{V}}}^{-1}\,\underline{\mathbf{u}}$ which approaches the CRLB depending on the functional form of $\xi[k]$. For example, for hyperbolic chirps, the variance of the estimator approaches the CRLB for small k_0 or large N. Overall, the estimator is very simple and efficient to compute.

3.3 Maximum Likelihood Estimator

The prevailing maximum likelihood estimation (MLE) technique requires the maximization of the probability density function of the data over the parameters' allowable range, $|c| < \infty$, $|\eta| < 0.5$

$$\max_{c,\eta} \sum_{k=k_0}^{N+k_0-1} \frac{\sqrt{|\tau[k]|}}{\sigma_e^2}\mathrm{Re}\{X[k]e^{-j(2\pi c\xi[k]+2\pi\eta k)}\}\,.$$

The above maximization can also be carried out in the TF plane by maximizing the correlation of GCWDs in (10). For most $\xi[k]$, e.g. the hyperbolic one, the MLE does not exist in closed form and 2-D grid searches are required. The method is thus computationally intensive and sensitive to the grid points chosen although it is expected to work well at low SNRs. Newton-Raphson iterative procedures reduce the computational sensitivity but not the computational cost [15].

3.4 Estimation of Hyperbolic chirps

The estimation techniques discussed above can be applied to hyperbolic chirps simply by using the hyperbolic group delay $\tau(f) = 1/f$ and $\xi(b) = \ln b$. For this simplified case, we analyze the performance of the estimators using simulations. The phase unwrapping plus linear regression of the phase data estimator in (18) is unbiased and attains the CRLB at SNRs higher than 12 dB, and it is very simple to implement. However, the constraints on the hyperbolic parameter c (15)-(17) limit it to only small values, $|c| < 2$. In Figure 1, we plot the mean square error (MSE) in estimating c for a hyperbolic chirp with true value $c = 0.1$ using this method and we compare it to the CRLB. Both are plotted as functions of SNR. In the same figure, the MLE performance is also shown to attain the CRLB at low SNRs, i.e. higher than -1 dB. For the MLE, we used numerical solutions based on a coarse 2-D grid

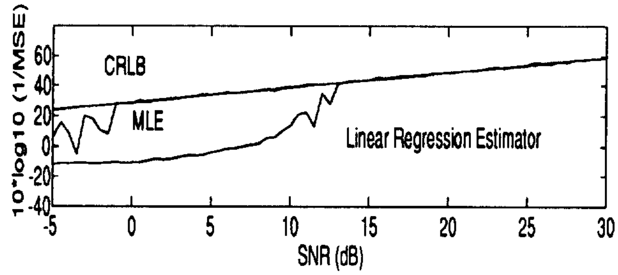

Figure 1: Hyperbolic chirp parameter c performance for $k_0 = 1$, $N = 30$, $c = 0.1$, and $\eta = 0.1$.

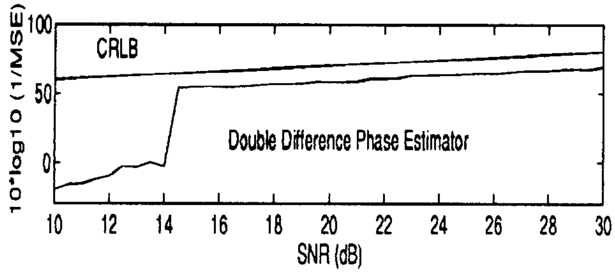

Figure 2: Hyperbolic chirp parameter c performance for $k_0 = 5$, $N = 300$, $c = 15$, and $\eta = 0.1$.

search to find initial guesses to a subsequent Newton-Raphson iteration [14]. The MLE performs well at low SNRs at the expense of high computational intensity. Note that similar results are obtained for the estimation of the normalized initial time parameter $|\eta| < 0.5$.

When we want to avoid the constraints that limit c to small values, or we only want to estimate c and not η, then the double difference phase estimator in (21) can be used. The performance of the estimator is shown in Figure 2 for a hyperbolic chirp with true value $c = 15$, and compared to the CRLB. The estimator is unbiased and approaches the CRLB for SNRs higher than 15 dB. Note that although it does not attain the CRLB like the phase unwrapping plus linear regression estimator, it approaches it for large N.

4 CONCLUSION

Generalized chirps are important nonstationary signals with dispersive law group delay characteristics. In this paper, we proposed the detection of generalized chirps in the presence of noise using generalized time-shift covariant QTFRs such as the generalized time-shift version of the Wigner distribution (GCWD). The optimal detection statistic reduces to a simple integration in the time-frequency plane of the GCWD of the observation signal along the generalized chirp's group delay. When the generalized chirp parameters are not known a priori, we proposed their estimation using various techniques such as maximum likelihood and phase unwrapping plus linear regression. For the special case of hyperbolic chirps, maximum likehood estimators are unbiased and attain the CRLB at low SNRs but at the cost of high computational intensity. The phase unwrapping plus linear regression estimators are unbiased and attain the CRLB at high SNRs and are very simple to compute. When a priori knowledge calls for high values of the generalized parameter c, then the double difference phase estimator can be used instead. It is also simple to implement and unbiased and approaches the CRLB at high SNRs.

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