On the Use of Operators versus Warpings versus Axiomatic Definitions of New Time-Frequency (Operator) Representations *

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ABSTRACT

The purpose of this paper is to give an overview of three techniques for deriving new time-frequency-scale representations, emphasizing their relative advantages and disadvantages. The three techniques considered are Cohen's generalized characteristic function method, the axiomatic approach, and unitary transformation methods.

1 INTRODUCTION

Within the past few years, a large number of new time-frequency (TFR), time-scale (TSR), and general $\mathcal{A}-\mathcal{B}$ "energy" representations have been developed using different and seemingly unrelated theoretical techniques. Their common theme is an attempt to map a one dimensional signal into a multi-dimensional representation in order to analyze the time-varying characteristics of non-stationary signals. Many were derived using one of the following three theoretical approaches: (1) the use of phase space operators to calculate characteristic functions, (2) axiomatic approaches where certain properties are deemed "desirable" and then the most general class of signal representations which achieves these properties is derived, and (3) unitary transformations or warpings on known TFRs and TSRs. The purpose of this paper is to give an overview of these approaches emphasizing their relative advantages and disadvantages.

1.1 Early Time-Frequency Representations

Since the Fourier Transform of the signal $s(t)$,

$$S(f) \;=\; \int s(t)e^{-j2\pi ft}dt = (\mathbb{F}s)(f), \qquad (1)$$

implicitly assumes that the signal's spectral content is not changing with time, there is a need for time-varying spectral representations for non-stationary signals. Many of the first mixed TFRs, e.g. Wigner-Ville, Gabor, spectrogram, Woodward Ambiguity function, etc.[1], were designed for specific applications. However, recent developments in TFRs and TSRs have focused on deriving large classes of quasi-distributions that satisfy one or more desirable properties and provide a unifying framework for earlier, application specific distributions.

1.2 Cohen-Class

One of the first TFR generalizations was Cohen's class of time-frequency representations.

$$C_S(t,f;\Psi_C)=\iiint\Psi_C(\tau,\nu;s)s\left(u+\frac{\tau}{2}\right)s^*\left(u-\frac{\tau}{2}\right) e^{j2\pi\nu(t-u)}e^{-j2\pi f\tau}d\tau d\nu du \qquad (2)$$

By changing the "kernel" $\Psi_C(\tau,\nu;s)$, a theoretically infinite number of TFRs are possible. If the kernel is signal independent, then the resulting bilinear TFRs correspond to the class of all quasi-distributions which are covariant to time and frequency shifts. Kernel constraints which guarantee other important distribution or signal analysis properties can be found in [5], [11], [1]. Positive TFRs that satisfy the marginals can be developed using signal dependent kernels[12]. Cohen has generalized his procedure for deriving TFRs to obtain time-"scale" representations or multi-dimensional representations of arbitrary physical variables [6].

	Table 1. Steps of Cohen's Method
1	$a \leftarrow (\mathbf{A}s)(x)$
2	$(\mathcal{A}_\alpha s)(x) = (e^{j2\pi\alpha\mathbf{A}}s)(x)$ $= \int\int e^{j2\pi\alpha}u_a^{\mathbf{A}*}(x')u_a^{\mathbf{A}}(x)s(x')dadx'$ $= (\sum_{n=0}^{\infty}\frac{(j2\pi\alpha)^n}{n!}\mathbf{A}^n s)(x)$
3	$M_s(\alpha) = < e^{j2\pi\alpha a} > = \int s^*(x)(e^{j2\pi\alpha\mathbf{A}}s)(x)dx$
4	$C_s(a;\Psi) = \int\Psi(\alpha;s)M_s(\alpha)e^{-j2\pi\alpha a}d\alpha$

Cohen's procedure uses Hermitian operators to find the characteristic function of quasi-distributions. It involves four main steps summarized in Table 1. First, an Hermitian operator $\mathbf{A}$ is found for each physical variable to be analyzed. Second, the unitary, exponential form, $\mathcal{A}_\alpha = e^{j2\pi\alpha\mathbf{A}}$ of this Hermitian oper-

*Funding in part by ONR grant N00014-92-J-1499

1058-6393/95 $4.00 © 1995 IEEE

ator is formed and its effect on the signal is evaluated using either a Taylor series expansion, evaluating the eigenequations of the Hermitian operator, i.e. $\mathbf{A}u_a^{\mathbf{A}}(x) = au_a^{\mathbf{A}}(x)$, or simplifying commutator rules. Examples of common Hermitian and associated Unitary operators are shown in the first two columns of Table 2. The eigenfunctions of the Hermitian operator $\mathbf{A}$ (and also of the corresponding $\mathcal{A}$) are given in the third column. The third step evaluates the characteristic function. Cohen shows that this equivalent to "sandwiching" the corresponding exponential operator between the signal and its conjugate in the integral equation in step 3. Different orderings or correspondence rules of the exponential operators give rise to different distributions, as is summarized in Table 3. Cohen postulated that the class of quasi-distributions resulting from all possible correspondence rules can be represented by multiplying the characteristic function with a kernel. Further, if the kernel is equal to one along its axes, then the marginal distributions, e.g. time, $|s(t)|^2$, frequency, $|S(f)|^2$, scale, $|(\mathbb{F}_{\mathbf{C}}s)(c)|^2$, and inverse frequency, $\frac{f_0}{r^2}|S(\frac{f_0}{r})|^2$, etc., are preserved. The final step takes the Fourier transform of the characteristic function multiplied by the kernel function Ψ to get the quasi-distribution function $C_s(\alpha;\Psi)$.

The advantages of Cohen's operator approach to characteristic function computation are the following: (1) It is a very general approach which unifies many previously derived TFRs and TSRs, as indicated in Table 3. (2) The four step procedure is the same regardless of the type or number of physical variables involved. (3) Simple constraints on the kernel guarantee that many desirable distribution properties are satisfied. (5) Many of the resulting TFRs can be computed using Fast Fourier transform algorithms. The disadvantages are: (1) One needs a working knowledge of operator theory, eigenfunction analysis, and commutator theory in order to evaluate the effect of the exponential operator on the signal in steps 2-3. In the multi-dimensional quasi-distributions, the argument of this exponential operator is the sum of several Hermitian operators. If the commutator of these operators, i.e. $[\mathbf{A},\mathbf{B}]=\mathbf{AB}-\mathbf{BA}$, is not a linear functional of the first operator, $\mathbf{A}$, then evaluation of steps 2-3 is difficult. (2) Although, in theory, the method works for all operators, in practice, only a few such operators, e.g. see Table 2, have been developed. (3) Each new correspondence rule (excluding dual operations in time and frequency) requires recalculation of all equations. (5) It is not clear which correspondence rule to select if one wants a particular type of distribution to result, so distribution synthesis (see axiomatic section) is difficult. (6) Not all time-varying representations, e.g. short time auto-regressive representations, can be put in this form. Further, Baraniuk has shown that Cohen's method does not generate all covariant and/or invariant forms of quasi-distributions [10].

1.3 Axiomatic Approach

The second method of generating large classes of TFRs and TSRs is the axiomatic approach, which corresponds to distribution synthesis. That is, find all TFRs that are covariant to a given set of signal operations, summarized below.

$$Y(f)=fcn(S(f);\underline{\alpha})\Rightarrow$$
$$T_y(t,f)=T_s(g_1(t,f;\underline{\alpha}),g_2(t,f;\underline{\alpha})) \quad (3)$$

Hence, if the signal $S(f)$ is changed in a certain way to form $Y(f)$, then the TFR of $Y(f)$ must also be related to the TFR of the original signal $S(f)$ in a pre-specified way. The synthesis formulation in eq. (3) was used to derive the affine [7], [9], shift-covariant [5], [11], Hyperbolic [2] and Power classes [8] of TFRs summarized in Table 4. For example, the first two columns of the first row reveal that if a signal $S(f)$ undergoes a scale change and a time-shift, then any TFR from the Affine class must undergo the same scale change and the same time shift. Similarly, hyperbolic class TFRs and the Bertrand $k=0$ distributions are covariant to scale changes and dispersive hyperbolic group delay changes in the signal.

Many of the TFR classes can be derived axiomatically by utilizing the fact that bilinear TFRs and TSRs can be formulated as the integral equation,

$$T_s(t,f)=\int\int k_T(t,f;f_1,f_2)S(f_1)S^*(f_2)df_1df_2 \quad (4)$$

involving a quadratic function of the signal $S(f)$ and a four-dimensional kernel, k_T. The two step axiomatic approach uses the covariance structure imposed in eq. (3) to simplify the kernel in (4). For example, for the hyperbolic class of TFRs, substituting (4) in both $T_y(t,f)$ and $T_s(g_1(t,f;\underline{\alpha}),g_2(t,f;\underline{\alpha}))$ in (3) with $\underline{\alpha}=[a\ c]$ results in the following two equations which must be equal when $Y(f)=\frac{1}{\sqrt{|a|}}S(\frac{f}{a})e^{-j2\pi c\ln f}$.

$$T_y(t,f)=\int\int k_T(t,f;\beta_1\beta_2)Y(\beta_1)Y^*(\beta_2)d\beta_1d\beta_2$$
$$=|a|\int\int k_T(t,f;af_1,af_2)e^{-j2\pi c\ln(f_1/f_2)}S(f_1)S^*(f_2)df_1df_2$$
$$T_s(a(t-\frac{c}{f}),\frac{f}{a})=\int\int k_T(a(t-\frac{c}{f}),\frac{f}{a};f_1f_2)S(f_1)S^*(f_2)df_1df_2$$

These equations are equal if

$$|a|k_T(t,f;af_1,af_2)e^{-j2\pi c\ln(f_1/f_2)}=k_T(a(t-\frac{c}{f}),\frac{f}{a};f_1f_2)$$

holds for any (t,f). Selecting $t=c/f$ and $f=a$, the kernel k_T simplifies to the two-dimensional kernel,

$$k_T(t,f;f_1,f_2)=\frac{1}{|f|}\Gamma(\frac{f_1}{f},\frac{f_2}{f})e^{j2\pi tf\ln\frac{f_1}{f_2}}$$

which indicates that the general expression of the Hyperbolic Class in Table 4 can be evaluated using Mellin transforms [2]. Hence, for the axiomatic classes in Table 4, the four dimensional kernel in (4) simplifies to a two-dimensional kernel whenever two covariance properties are imposed, e.g. affine, shift covariant, power classes, or a one dimensional kernel when three properties are imposed, e.g. Bertrand and shift-scale covariant classes.

The advantages of the axiomatic approach are that it is ideally suited for distribution synthesis, which occurs when particular applications demand that a TFR or TSR satisfy certain covariance properties. For example, shift covariance is important in speech processing and pattern recognition. In most instances, only knowledge of calculus is required to use the axiomatic approach. A variety of desirable distribution properties can be guaranteed by placing relatively simple constraints on the kernel Γ. The disadvantages are that for each new set of covariance properties, new integral equations must be computed. These integrals can be tedious, obscuring the underlying structure of the resulting representation. Fast algorithms do not always exist for the more complicated integral equations in the axiomatic approach.

1.4 Unitary Warpings

The third method of generalizing large classes of TFRs and TSRs involves the use of unitarily equivalent operators, $\mathcal{A}$ and $\tilde{\mathcal{A}}=\mathcal{U}^{-1}\mathcal{A}\mathcal{U}$, where $\mathcal{U}$ is a unitary operator [10]. Unitary warping of analysis windows and mother wavelets have been used to transform the rectangular tiling pattern of the time frequency plane, which is characteristic of traditional Short-time Fourier transform and wavelet transform techniques, into non-rectangular tilings corresponding to the fan, chevron, bowtie, chirplet and metaplectic transformations [13],[4]. The basis functions are warped to better match important signal characteristics. Baraniuk uses unitarily equivalent operators of $\mathcal{E},\mathcal{S},\mathcal{C}$ in Table 2 to provide insight into the characteristic function method of Cohen and to provide a unifying theory combining many of the TFRs and TSRs derived axiomatically. His procedure uses three conceptual steps. First, pre-warp the signal by a unitary operator $\mathcal{U}$. Second, input the warped signal into a conventional Cohen class TFR to form the "pre-warped" $\mathcal{U}$-Cohen class. Third, post-warp the $\mathcal{U}$-Cohen class of warped TFRs by the operator $\mathcal{V}$, so that signals will occur in their correct time-frequency location. He refers to the final output as the doubly warped $\mathcal{VU}$-Cohen class. A similar procedure can be used to form the $\mathcal{VU}$-Affine class. This procedure leads to some very intuitive and useful results. Since Cohen's class is covariant to the time and frequency shift operators $\mathcal{S}_\tau,\mathcal{E}_\nu$ formed from the Hermitian operators $\mathbf{T},\mathbf{F}$ in Table 2, then the $\mathcal{U}$-Cohen class is covariant to their unitarily equivalent operators $\tilde{\mathcal{E}}_\alpha=\mathcal{U}^{-1}\mathcal{E}_\alpha\mathcal{U}$, and $\tilde{\mathcal{S}}_\beta=\mathcal{U}^{-1}\mathcal{S}_\beta\mathcal{U}$. Since the Wigner Distribution maps tones ($u_\nu^{\mathbf{F}}(t)$ in Table 2), impulses ($u_\tau^{\mathbf{T}}(t)$) and linear FM chirps to straight lines in the time-frequency plane, then the $\mathcal{U}$-Wigner Distribution will map $(\mathcal{U}^{-1}u_\nu^{\mathbf{F}})(t)$, $(\mathcal{U}^{-1}u_\tau^{\mathbf{T}})(t)$ and chirps pre-warped by $\mathcal{U}^{-1}$ to lines in the warped time-frequency plane. If the kernel Ψ in (2) is equal to one along its axes, then the marginals of the $\mathcal{U}$-Cohen class will map to $|(\mathcal{U}s)(a)|^2$ and $|(\mathbb{F}\mathcal{U}s)(b)|^2$.

The advantages of Baraniuk's $\mathcal{U}$-Cohen class method is that it provides a powerful and succinct framework for unifying and interpreting many axiomatically derived TFRs. It generalizes to Cohen's distribution of arbitrary operators $\mathcal{A},\mathcal{B}$ as long as they are unitarily equivalent to $\mathcal{E},\mathcal{S}$ in Table 2 [6]. Further, many of the warped TFRs can be computed using the conventional TFR algorithms provided that one inputs the pre-warped signal, $(\mathcal{U}s)(x)$, i.e. $C_{\mathcal{U}s}(a,b) = \mathbb{F}_{\alpha\mapsto a}\mathbb{F}_{\beta\mapsto b}[\Psi_C(\alpha,\beta)M_{\mathcal{U}s}(\alpha,\beta)]$. Consequently, several of the kernel design techniques and fast algorithm implementations of Cohen class TFRs can be used to compute the new, $\mathcal{U}$-Cohen distributions. The method provides insight on how to design $\mathcal{U}$-Cohen class TFRs to "match" the eigenfunctions listed in Table 2, in order to achieve perfect concentration in the warped time-frequency plane for warped tones, warped impulses, warped chirps or any signal class with a one-to-one instantaneous frequency or group delay function. Simple kernel constraints will insure that desirable properties hold and that the warped marginals, which correspond to the squared magnitude of the generalized Fourier transforms in the last column of Table 2, are preserved. The disadvantages are that one needs a good command of some very advanced mathematical tools to understand and to utilize the warped formulations. (2) One cannot warp time and frequency variables independently. Hence, an intuitive or useful warping on the first distribution variable may result in a difficult to interpret warping of the second variable. In principle, an infinite number of choices for the warping operator $\mathcal{U}$ are possible; in practice, only a few, some of which are given in Table 5, have been shown to map both time, frequency, or scale into useful variables. (3) One cannot generate all TFRs and TSRs from this method. For example, Cohen's shift covariant class cannot be unitarily warped to obtain the affine class. (4) The warped distribution often has parameters which do not correspond to time or frequency or scale. If absolute time-frequency location is required in signal analysis/detection problems, then a "post-warping" must be performed in order to change the warped axes back to true time and frequency. Finally, in practice, it may be difficult to "pre-warp" real-world signals in arbitrary fashion; many such warpings necessitate sampling the signal arbitrarily quickly in a non-uniform fashion, which makes investigation of aliasing issues difficult.

2 Acknowledgements

The author would like to acknowledge helpful conversations with A. Papandreou, R. Baraniuk, and F. Hlawatsch and the help given by R. Murray in preparing this manuscript.

$\mathcal{U}$	$\mathcal{A}$	$\tilde{\mathcal{A}}=\mathcal{U}^{-1}\mathcal{A}\mathcal{U}$
$\mathbb{F}$	$\mathbf{T}$	$\mathbf{F}$
	$\mathbf{F}$	$-\mathbf{T}$
	$\mathbf{C}$	$-\mathbf{C}$
	$\mathcal{E}_\alpha$	$\mathcal{S}_\alpha$
	$\mathcal{S}_\alpha$	$\mathcal{E}_{-\alpha}$
	$\mathcal{C}_\alpha$	$\mathcal{C}_{-\alpha}$
	$\mathcal{U}_{hyp}$	$\mathcal{U}_{log}$
$\mathcal{C}_\beta$	$\mathcal{E}_\alpha$	$\mathcal{E}_{\alpha e^{-\beta}}$
	$\mathcal{S}_\alpha$	$\mathcal{S}_{e^{\beta}\alpha}$
	$\mathcal{C}_\alpha$	$\mathcal{C}_\alpha$
$\mathcal{U}_{hyp}$	$\mathbf{T}$	$\ln \mathbf{T}$
	$\mathbf{F}$	$\mathbf{C}$
	$\mathcal{E}_\alpha$	$\mathcal{D}_\alpha$
	$\mathcal{S}_\alpha$	$\mathcal{C}_\alpha$
$\mathbb{F}^{-1}\mathcal{U}_{hyp}\mathbb{F}$	$\mathbf{T}$	$\mathbf{C}$
	$\mathbf{F}$	$\mathbb{F}^{-1} \ln \mathbf{T}\, \mathbb{F}$
	$\mathcal{E}_\alpha$	$\mathcal{C}_\alpha$
	$\mathcal{S}_\alpha$	$\mathbb{F}^{-1}\mathcal{D}_\alpha\mathbb{F}$

Table 5. Useful Unitarily equivalent operators. Here, $\mathbf{T}, \mathbf{F}, \mathbf{C}, \mathcal{E}, \mathcal{S}, \mathcal{C}$ are defined in Table 2 and $\mathbb{F}$ in eq. (1). $(\mathcal{U}_{hyp}s)(x) = e^{x/2}s(e^x)$, $(\mathcal{U}_{hyp}^{-1}s)(x) = \frac{1}{\sqrt{x}}s(\ln x)$, $x > 0$, $(\mathcal{U}_{log}s)(x) = \mathbb{F}_{f\mapsto x}[e^{f/2}S(e^f)]$, and $(\mathcal{D}_\alpha s)(x) = e^{j2\pi\alpha \ln x}s(x)$. Note that $\mathcal{U}_{hyp}$ warps $\mathbf{T}$ and $\mathbf{F}$ to log time and scale.

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Ordering	$C_s(t, f, c; \Psi)$
$e^{j2\pi\tau\mathbf{F}}$	$\lvert S(f)\rvert^2$ Power Spectral Density
$e^{j2\pi\nu\mathbf{T}}$	$\lvert s(t)\rvert^2$ Instant. Signal Energy
$e^{j2\pi\sigma\mathbf{C}}$	$\lvert\int s(\tau)e^{-j2\pi c\ln\tau}\frac{d\tau}{\sqrt{\tau}}\rvert^2$ $\lvert$Mellin Transform$\rvert^2$
$e^{j\pi\sigma\mathbf{C}}e^{j2\pi\nu\mathbf{T}}e^{j\pi\sigma\mathbf{C}}$	$\int s(te^{\frac{\sigma}{2}})s^*(te^{-\frac{\sigma}{2}})e^{-j2\pi\sigma c}d\sigma$ Scale Invariant Wigner D.
$e^{j\pi\sigma\mathbf{C}}e^{j2\pi\tau\mathbf{F}}e^{j\pi\sigma\mathbf{C}}$	$\int S(fe^{-\frac{\sigma}{2}})S^*(fe^{\frac{\sigma}{2}})e^{j2\pi\sigma c}dc$ Altes Distribution
$e^{j2\pi(\nu\mathbf{T}+\tau\mathbf{F})}$	$\int s(t+\frac{\tau}{2})s^*(t-\frac{\tau}{2})e^{-j2\pi\tau f}d\tau$ Wigner Distribution (WD)
$e^{j2\pi(\tau\mathbf{F}+\sigma\mathbf{C})}$	$\int \mu_0(\sigma)S(fe^{\sigma/2}\mu_0(\sigma))$ $S^*(fe^{-\sigma/2}\mu_0(\sigma))e^{j2\pi\sigma c}d\sigma$ Bertrand P_0 Distrib.
$e^{j2\pi(\nu\mathbf{T}+\sigma\mathbf{C})}$	$\int s(te^{\sigma/2}\mu_0(\sigma))s^*(te^{-\sigma/2}\cdot$ $\mu_0(\sigma))\mu_0(\sigma)e^{-j2\pi\sigma c}d\sigma$ dual Bertrand P_0 Distrib.
$e^{j2\pi\nu\mathbf{T}}e^{j2\pi\tau\mathbf{F}}$	$s^*(t)S(f)e^{j2\pi ft}$ (Rihaczek Distribution)*
$e^{j2\pi\tau\mathbf{F}}e^{j2\pi\nu\mathbf{T}}$	$s(t)S^*(f)e^{-j2\pi ft}$ Rihaczek Distribution
$(e^{j2\pi\nu\mathbf{T}}e^{j2\pi\tau\mathbf{F}}+e^{j2\pi\tau\mathbf{F}}e^{j2\pi\nu\mathbf{T}})/2$	$Re\{s^*(t)S(f)e^{j2\pi ft}\}$ Ackroyd Distribution Margineau-Hill Distrib.

Table 3. Orderings or Correspondence Rules and associated quasi-distributions $C_s(t, f, c; \Psi)$. $\mu_0(u) = [u/2]/[\sinh(u/2)]$ and the kernel $\Psi(\tau, \nu, \sigma) = 1$.

$Y(f)$	$T_Y(t,f)$	$T_S(t,f;\Gamma)$
$\frac{1}{\sqrt{\lvert a\rvert}}S(\frac{f}{a})e^{-j2\pi f t_0}$	$T_S(a(t-t_0),\frac{f}{a})$	**Affine Class** $\frac{1}{\lvert f\rvert}\int\int\Gamma\left(\frac{f_1}{f},\frac{f_2}{f}\right)S(f_1)S^*(f_2)e^{j2\pi t(f_1-f_2)}df_1df_2$
$\frac{1}{\sqrt{\lvert a\rvert}}S(\frac{f}{a})e^{-j2\pi(ft_0+c\theta_k(f))}$ with $\theta_k(f)=\begin{cases}\ln f, & k=0\\ f\ln f, & k=1\\ f^k, & k\neq 0,1\end{cases}$	$T_s(a[t-t_0-c\theta_k'(f)],\frac{f}{a})$ $\theta_k'(f)=\frac{d}{df}\theta_k(f)$	**Bertrand Affine $\mathbf{P_k}$ Distributions** $f\int S(f\lambda_k(u))S^*(f\lambda_k(-u))\Gamma_k(u)e^{j2\pi tf(\lambda_k(u)-\lambda_k(-u))}du,\ f>0$ with $\lambda_0(u)=\frac{u/2e^{u/2}}{\sinh(u/2)},\ \lambda_1(u)=\left[1+\frac{ue^{-u}}{e^{-u}-1}\right],$ $\lambda_k(u)=\left[k\frac{e^{-u}-1}{e^{-ku}-1}\right]^{\frac{1}{k-1}},k\neq 0,1$ and $\Gamma_k(u)=\Gamma_k(-u)>0$
$S(f-f_0)e^{-j2\pi(f-f_0)t_0}$	$T_s(t-t_0,f-f_0)$	**Shift Covariant Class** $\int\int\Gamma(f-f',\nu)S(f'+\frac{\nu}{2})S^*(f'-\frac{\nu}{2})e^{j2\pi\nu t}df'd\nu$
$\frac{1}{\sqrt{\lvert a\rvert}}S(\frac{f}{a})e^{-j2\pi c\ln f}$	$T_S(a(t-\frac{c}{f}),\frac{f}{a})$	**Hyperbolic Class** $\frac{1}{f}\int_0^\infty\int_0^\infty\Gamma(\frac{f_1}{f},\frac{f_2}{f})S(f_1)S^*(f_2)e^{j2\pi tf\ln(f_1/f_2)}df_1df_2,\ \ f>0$
$\frac{1}{\sqrt{\lvert a\rvert}}S(\frac{f}{a})e^{-j2\pi c\xi_\kappa(f)}$, $\xi_\kappa(f)=\lvert f\rvert^\kappa\mathrm{sgn}(f),\ \kappa\neq 0$	$T_S(a(t-c\tau_\kappa(f)),\frac{f}{a})$ $\tau_\kappa(f)=\frac{d}{df}\xi_\kappa(f)$	**κ^{th} Power Class** $\frac{1}{\lvert f\rvert}\int\int\Gamma^{(\kappa)}\left(\frac{f_1}{f},\frac{f_2}{f}\right)e^{j2\pi\frac{t[\xi_\kappa(f_1)-\xi_\kappa(f_2)]}{\tau_\kappa(f)}}S(f_1)S^*(f_2)df_1df_2$ $\Gamma^{(\kappa)}(b_1,b_2)=\lvert\tau_\kappa(\sqrt{\lvert b_1b_2\rvert})\rvert\tilde{\Gamma}\left(\frac{\xi_\kappa(b_1)+\xi_\kappa(b_2)}{2},\xi_\kappa(b_1)-\xi_\kappa(b_2)\right),\ \kappa\neq 0$
$\frac{1}{\sqrt{\lvert a\rvert}}S(\frac{f-f_0}{a})e^{-j2\pi(f-f_0)t_0}$	$T_S(a(t-t_0),\frac{f-f_0}{a})$	**Shift – Scale Covariant Class** $\int\int\frac{1}{\lvert\nu\rvert}\Gamma\left(-\left[\frac{f-f'}{\nu}\right]\right)S(f'+\nu/2)S^*(f'-\nu/2)e^{j2\pi t\nu}df'd\nu$

Table 4. Classes of Time-Frequency Representations derived using Axiomatic Techniques

Hermitian Operator $(\mathbf{A}s)(x)$	Unitary Operator $(\mathcal{A}_\alpha s)(x)=(e^{j2\pi\alpha\mathbf{A}}s)(x)$	Eigenfunction $u_a^{\mathbf{A}}(x)=u_a^{\mathcal{A}}(x)$	Generalized Fourier Trans. $(\mathbb{F}_{\mathbf{A}}s)(a)=<s,u_a^{\mathbf{A}}>$ $=\int s(x)[u_a^{\mathbf{A}}(x)]^*dx$
$(\mathbf{T}s)(x)=xs(x)$	$(\mathcal{E}_\alpha s)(x)=(e^{j2\pi\alpha\mathbf{T}}s)(x)$ $=(e^{j2\pi\alpha x}s)(x)$	$\delta(x-a)$	$s(a)$
$(\mathbf{F}s)(x)=\frac{1}{2\pi j}\frac{d}{dx}s(x)$	$(\mathcal{S}_\alpha s)(x)=(e^{j2\pi\alpha\mathbf{F}}s)(x)$ $=s(x+\alpha)$	$e^{j2\pi ax}$	$S(a)$
$(\mathbf{C}s)(x)$ $=(\frac{1}{2}[\mathbf{TF}+\mathbf{FT}]s)(x)$	$(\mathcal{C}_\alpha s)(x)=(e^{j2\pi\alpha\mathbf{C}}s)(x)$ $=e^{\alpha/2}s(e^\alpha x)$	$\frac{1}{\sqrt{x}}e^{j2\pi a\ln x},\ \ x>0$	$\int_0^\infty s(x)e^{-j2\pi a\ln x}\frac{dx}{\sqrt{x}}$
$(\mathbf{R}s)(x)=(\frac{f_0}{\mathbf{F}}s)(x)$		$\frac{\sqrt{f_0}}{a}e^{j2\pi f_0x/a}$	$\frac{\sqrt{f_0}}{a}S(\frac{f_0}{a})$
$([\nu\mathbf{T}+\tau\mathbf{F}]s)(x)$	$(e^{j2\pi\nu\mathbf{T}+j2\pi\tau\mathbf{F}}s)(x)=$ $e^{j\pi\nu\tau}e^{j2\pi\nu x}s(x+\tau)$	$\frac{1}{\sqrt{\tau}}e^{j2\pi(ax-\nu x^2/2)/\tau}$	$\mathbb{F}_{x\mapsto a}$ $\sqrt{\tau}s(\tau x)e^{j\pi\nu\tau x^2}$
$([\nu\mathbf{T}+\kappa\mathbf{R}]s)(x)$	$(e^{j2\pi\nu\mathbf{T}+j2\pi\kappa\mathbf{R}}s)(x)=\mathbb{F}^{-1}_{f\mapsto x}$ $e^{j(2\pi f_0\kappa/\nu)\ln[f/(f-\nu)]}S(f-\nu)$	$\mathbb{F}^{-1}_{f\mapsto x}$ $\frac{1}{\sqrt{\nu}}e^{-j2\pi(af-f_0\kappa\ln\lvert f\rvert)/\nu}$	$\mathbb{F}^{-1}_{f\mapsto a}$ $\sqrt{\nu}S(\nu f)e^{-j2\pi\kappa(f_0/\nu)\ln\lvert\nu f\rvert}$
	$(\mathcal{L}s)(x)=\sqrt{\lvert\frac{d}{dx}w(x)\rvert}\ s(w(x))$	$\sqrt{\lvert\frac{d}{dx}m(x)\rvert}\ \delta(m(x)-a)$ where $m(x)=w^{-1}(x)$	$\lvert\frac{d}{dx}m(x)\rvert^{-1/2}s(x)\Big\rvert_{x=w(a)}$
	$(\tilde{\mathcal{A}}s)(x)=(\mathcal{U}^{-1}\mathcal{A}\mathcal{U}s)(x)$	$(\mathcal{U}^{-1}u_a^{\mathbf{A}})(x)$	$(\mathbb{F}_{\tilde{\mathbf{A}}}s)(a)=(\mathbb{F}_{\mathbf{A}}\mathcal{U}s)(a)$ $=<s,\mathcal{U}^{-1}u_a^{\mathbf{A}}>$

Table 2. Hermitian and Unitary Operators, their associated eigenfunctions, and generalized Fourier Transform