A Stochastic One-Dimensional Image Model Based on Occluding Object Images

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Abstract

This paper provides new insights into the formation of one-dimensional (line-scan) image autocorrelation functions. We model a line scan as a composition of individual object-images that have random positions, widths and intensities and that occlude one another. We derive the autocorrelation function of this model as a function of object-image width and intensity distributions. We show that any assumption regarding the form of the autocorrelation function places a constraint on object-image width and intensity distributions and we derive the object-image width distribution associated with the widely used symmetric-exponential autocovariance model.

1: Introduction

The function $\exp\{-\alpha \mid x_2-x_1 \mid \}$ and its extensions to two dimensions $\sigma^2\exp\left\{-\alpha\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\right\}$ and $\sigma^2\exp\{-\alpha \mid x_2-x_1 \mid -\beta|y_2-y_1|\}$ have been used for decades to model one and two-dimensional autocovariance functions of images [1]-[8]. These models have led to the introduction of image models that agree with the exponential form. The image models break into two classes: first-order autoregressive models [6]-[7] and models consisting of piecewise constant patches of image intensity [8]-[9]. Schreiber [8] gave the following explanation for the approximate exponential form: Consider a one-dimensional image consisting of a number of patches having various uniform intensities and widths. The autocorrelation function of one patch is a triangle having height $\mathbb{I}^2 w$ and base $2w$ where $\mathbb{I}$ is the patch's intensity and w is its width. The autocorrelation function of the one-dimensional image is the sum of a large number of such triangles having various heights and widths. This sum can be approximated by an exponential.

In this paper we represent a one-dimensional image (a line scan) as a composition of stochastic object images created by the image formation process; hence the object images occlude portions of both background and other object images. We derive the autocovariance function of this image model as a function of object-image width and intensity distributions. Since the image model captures the phenomenon of occlusion, it provides new insights into the covariance structure of images. Extension of the analysis of this paper to two dimensions is possible. However, extension of the results to two dimensions using assumptions of autocovariance separability or isotropy is approximate and should be done only with great care.

We begin by defining a *scene image*, $s_n(x)$, $x \in [-L, L]$, composed of a stochastic *background image*, $b(x)$, and n randomly placed stochastic *object images*, $o_i(x-p_i)$, $i = 1, 2, \cdots, n$, where p_i is a random one-dimensional position vector for the i^{th} object image. We assume that $o_i(x)$ has region of support, $\mathcal{D}_i = [-w_i/2, w_i/2]$. Thus, w_i equals the width of object image $o_i(x)$ before it may be occluded by other object images. We account for occlusion by recursive generation of the scene image. At the i^{th} step of the recursion, $i = 1, 2, \cdots, n$, the scene image is given by

$$s_i(x) = \begin{cases} o_i(x-p_i) & \text{if } x \in \mathcal{D}_i + p_i \\ s_{i-1}(x) & \text{otherwise} \end{cases} \tag{1}$$

for $i = 1, 2, \cdots, n$ where $s_0(x) \equiv b(x)$ and $x \in [-L, L]$. Thus, at recursion step i, object image $o_i(x-p_i)$, is superimposed on the scene image $s_{i-1}(x)$ in such a way such that it occludes $s_{i-1}(x)$ for $x \in [p_i - w_i/2,\, p_i + w_i/2]$. We define our image model to be $s_n(x)$ in the limit $L \to \infty$ holding $n = \lambda 2L$ where λ is a constant. To write (1) more compactly we introduce the indicator function

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$$I(x;w) = \begin{cases} 1 & \text{for } x \in [-w/2, w/2] \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

and its complement

$$I^c(x;w_i) = 1 - I(x;w_i) \quad (3)$$

Since $o_i(x-p_i)I(x-p_i;w_i) \equiv o_i(x-p_i)$, (1) becomes

$$s_i(x) = o_i(x-p_i) + s_{i-1}(x)I^c(x-p_i;w_i) \quad (4)$$

where $i = 1, 2, \cdots, n$; $s_0(x) = b(x)$; and $x \in [-L, L]$.

2: Derivation

The derivation of the mean and autocovariance functions of $s_n(x)$ in the limit $L \to \infty$ with $n = \lambda 2L$ is outlined below.

2.1: Assumptions

The derivation is based on the following assumptions:

1. The object images are random processes of the form

$$o_i(x) = \mathbb{I}_i(x)I(x;w_i) \quad (5)$$

where the "intensity", $\mathbb{I}_i(x)$, is a sample function of a wide-sense stationary process, independent of the object-image width, w_i. Object image $o_i(x)$ and its center point p_i of (4) are statistically independent of $s_{i-1}(x)$ for $i = 1, 2, \ldots, n$..

2. The i-tuple $\underline{w}_i = (w_1, w_2, \cdots, w_i)$ has probability density function (pdf)

$$f_{\underline{w}_i}(\underline{W}_i) = \prod_{j=1}^{i} f_w(W_j) \quad (6)$$

for $i = 1, 2, \cdots n$, where $\underline{W}_i = (W_1, W_2, \ldots, W_i)$ and

$$f_w(W) = p_w(W \mid w < 2L) \quad (7)$$

where $\mid$ denotes "conditioned on". $p_w(W)$ is a pdf of a random variable, $w \in (0, \infty)$.

3. $\underline{p}_i = (p_1, p_2, \cdots, p_i)$ has conditional probability density function (pdf)

$$f_{\underline{p}|\underline{w}}(\underline{P}_i \mid \underline{W}_i) = \prod_{j=1}^{i} f_{p|w}(P_j \mid W_j) \quad (8)$$

for $i = 1, 2, \cdots, n$ where $\underline{P}_i = (P_1, P_2, \ldots, P_i)$ and

$$f_{p|w}(P \mid W) = \begin{cases} \frac{1}{2L+W} & \text{for } |P| \le L + 0.5W \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

The above assumptions of independent object-image intensities, widths, and uniformly-random center points are maximum-entropy assumptions.

2.2: Scene-Image Mean

We denote the mean, $E\{a(x)\}$, of a random process $a(x)$ by $\eta_a(x)$. The following formula for the conditional mean of $s_i(x)$ follows directly from (4) and the assumptions:

$$\eta_{s_i}(x \mid \underline{W}_i, \underline{P}_i) = \eta_o(x - P_i \mid W_i) + \eta_{s_{i-1}}(x \mid \underline{W}_{i-1}, \underline{P}_{i-1})I^c(x - P_i; W_i) \quad (10)$$

where $x \in [-L, L]$, $i = 1, 2, \cdots, n$, $\underline{W}_0$ and $\underline{P}_0$ are empty sets, and $\eta_{s_0}(x \mid \underline{W}_0, \underline{P}_0) \equiv \eta_b(x)$, After we remove the conditions from $\eta_{s_i}(x \mid \underline{W}_i, \underline{P}_i)$ we obtain,

$$\eta_{s_i}(x) = \eta_{\mathbb{I}}(1 - \xi) + \eta_{s_{i-1}}(x)\xi \quad (11)$$

where $\eta_{\mathbb{I}} = E\{\mathbb{I}(x)\}$ is constant and

$$\xi = \int_0^\infty \frac{2L}{2L + W} f_w(W)dW, \quad (12)$$

$x \in [-L, L]$ and $i = 1, 2, \cdots, n$. The solution to (11) is

$$\eta_{s_n}(x) = \eta_{\mathbb{I}}(1 - \xi)\sum_{i=1}^{n} \xi^{n-i} + \eta_b(x)\xi^n$$
$$= \eta_{\mathbb{I}}(1 - \xi^n) + \eta_b(x)\xi^n \quad (13)$$

The limiting form of $\eta_{s_n}(x)$ for $L \to \infty$ with $n = \lambda 2L$ is (see Appendix A)

$$\eta_s(x) = \eta_{\mathbb{I}}\,(1 - e^{-\lambda \overline{w}}) + \eta_b(x)e^{-\lambda \overline{w}} \quad (14)$$

$\forall x$ where

$$\overline{w} \triangleq E\{w\} = \int_0^\infty W p_w(W)dW \quad (15)$$

In the limit $L \to \infty$, the probability of having N object-image centers in an interval of length ℓ is Poisson: $P(N) = (\lambda\ell)^N exp\{-\lambda\ell\}/N!$ where λ equals the expected number of object-image centers per unit length [10]. The pdf of the distance, d, between adjacent object-image centers is exponential with $E\{d\} = 1/\lambda$ [10].

The following alternative expression for $\eta_s(x)$ can be found from elementary probability theory:

$$\eta_s(x) = \eta_b(x)P_b(0) + \eta_{\mathbb{I}}P_o(0) \tag{16}$$

where $P_b(0) \equiv \Pr\{B(x)\}$ is the probability of the event $B(x) = $ {background appears (is not occluded) at x} and $P_o(0) \equiv \Pr\{O(x)\} = 1 - P_b(0)$ is the probability of the event $O(x) = $ {an object-image appears at x}. It follows from (13) and (16) that

$$P_b(0) = e^{-\lambda\overline{w}} \tag{17}$$
$$P_o(0) = 1-e^{-\lambda\overline{w}} \tag{18}$$

Note that the background process, $b(x)$, is occluded with probability one in the limit $\lambda \to \infty$.

2.3: Scene-Image Autocovariance

We denote the autocovariance function of a process $a(x)$ by $K_a(x_1, x_2) = R_a(x_1, x_2) - \eta_a(x_1)\eta_a(x_2)$ where $R_a(x_1, x_2) = E\{a(x_1)a(x_2)\}$ is the autocorrelation function of $a(x)$. We temporarily assume that $\eta_{\mathbb{I}} = \eta_b(x) = 0$ and use a prime on s of (4) to indicate this assumption. The following formula for the autocovariance (or autocorrelation) function of $s_i'(x)$ is obtained directly from (4) and the assumptions:

$$\begin{aligned} K_{s_i'}(x_1, x_2 \mid \underline{W}_i, \underline{P}_i) &= K_o(x_1-P_i, x_2-P_i \mid W_i) \\ &+ [K_{s_{i-1}'}(x_1, x_2 \mid \underline{W}_{i-1}, \underline{P}_{i-1}) \times \\ &\quad I^c(x_1-P_i; W_i)\, I^c(x_2-P_i; W_i)] \end{aligned} \tag{19}$$

We will set $x_1 \equiv x$ and $x_2 \equiv x + \Delta$. After we remove the conditions from (19), we obtain, for $x, x+\Delta \in [-L, L]$:

$$K_{s_i'}(x, x+\Delta) = \overline{K}_o(\Delta) + K_{s_{i-1}'}(x, x+\Delta)\phi(\Delta) \tag{20a}$$

for $i = 1, 2, \cdots, n$ and

$$K_{s_0'}(x, x+\Delta) \equiv K_b(x, x+\Delta) \tag{20b}$$

where

$$\begin{aligned} \overline{K}_o(\Delta) &= E\{K_o(x_1-p_i, x_2-p_i \mid w_i)\} \\ &= K_{\mathbb{I}}(\Delta)[\phi(\Delta) - \theta(\Delta)] \end{aligned} \tag{21}$$

The functions $\phi(\Delta)$ and $\theta(\Delta)$ appearing in (20a) and (21) are defined by

$$\phi(\Delta) = \int_0^\infty \tfrac{2L-W}{2L+W}\left[1 + \tfrac{W}{2L-W}\mathbb{T}\left(\tfrac{\Delta}{W}\right)\right] f_w(W)dW \tag{22a}$$

and

$$\theta(\Delta) = \int_0^\infty \tfrac{2L-W}{2L+W} f_w(W)dW \tag{22b}$$

where

$$\mathbb{T}(x) = \begin{cases} 1 - x & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{22c}$$

The function $\mathbb{T}(\Delta/W)$ is analogous to Schreiber's triangle [8] mentioned in Section 1. The solution to (20) is

$$\begin{aligned} K_{s_n'}(x, x+\Delta) &= \overline{K}_o(\Delta)\sum_{i=1}^{n}\phi^{n-i}(\Delta) \\ &+ K_b(x, x+\Delta)\phi^n(\Delta) \\ &= K_{\mathbb{I}}(\Delta)\tfrac{\phi(\Delta)-\theta(\Delta)}{1-\phi(\Delta)}\left[1-\phi^n(\Delta)\right] + K_b(x, x+\Delta)\phi^n(\Delta) \end{aligned} \tag{23}$$

which becomes, in the limit $L \to \infty$ with $n = \lambda 2L$ (see Appendix A):

$$\begin{aligned} K_{s'}(x, x+\Delta) &= K_{\mathbb{I}}(\Delta)\left\{\tfrac{1-\psi(\Delta)}{1+\psi(\Delta)}\right\}\left\{1-e^{-\lambda\overline{w}[1+\psi(\Delta)]}\right\} \\ &+ K_b(x, x+\Delta)e^{-\lambda\overline{w}[1+\psi(\Delta)]} \end{aligned} \tag{24}$$

$\forall\, x$ where

$$\psi(\Delta) = \frac{1}{\overline{w}}\int_0^{|\Delta|} [1 - P_w(W)]dW \tag{25}$$

and

$$P_w(W) = \int_0^W p_w(\sigma)d\sigma \tag{26}$$

The following alternative expression for $K_{s_n'}(x, x+\Delta)$ can be found from elementary probability theory:

$$\begin{aligned} K_{s'}(x, x+\Delta) &= K_{\mathbb{I}}(\Delta)\, P_o(\Delta) \\ &+ K_b(x, x+\Delta)\, P_b(\Delta) \end{aligned} \tag{27}$$

where $P_b(\Delta) = \Pr\{B(x) \cap B(x+\Delta)\}$ and $P_o(\Delta) = \Pr\{O(x, x+\Delta)\}$ where the event $O(x, x+\Delta) = $ {parts of the *same* object are visible at x and $x+\Delta$}. It follows from (24) and (27) that

$$P_b(\Delta) = e^{-\lambda\overline{w}[1+\psi(\Delta)]} \quad (28)$$

$$P_o(\Delta) = \left\{\frac{1-\psi(\Delta)}{1+\psi(\Delta)}\right\}\left\{1-e^{-\lambda\overline{w}[1+\psi(\Delta)]}\right\} \quad (29)$$

The function $\psi(\Delta)$ of (25) is given by the shaded area shown in Figure 1 for $\Delta \geq 0$. $\psi(\Delta)$ is an even-symmetric function of Δ, monotonically increasing in $|\Delta|$ with $\psi(0) = 0$ and $\psi(\pm\infty) = 1$.

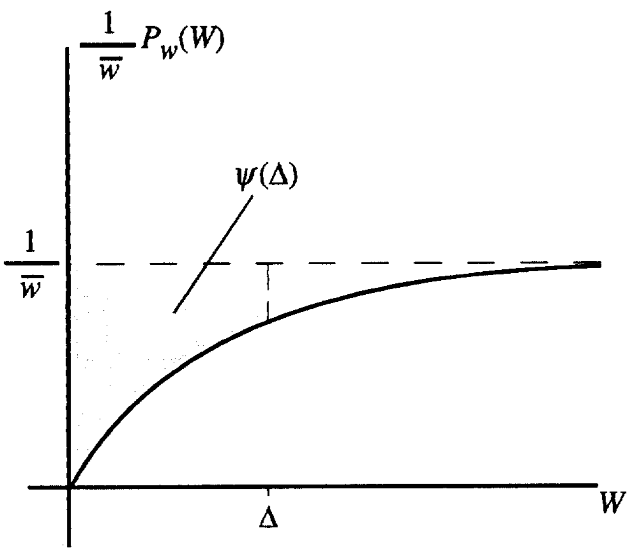

Figure 1. Interpretation of $\psi(\Delta)$

$P_b(\Delta)$ and $P_o(\Delta)$ of (28) and (29) are even symmetric functions of Δ, which decrease monotonically with $|\Delta|$ to $P_b(\pm\infty) = exp(-2\lambda\overline{w})$ and $P_o(\pm\infty) = 0$. $P_b(\Delta)$ decreases exponentially with λ while $P_o(\Delta)$ increases with λ. This behavior reflects the increasing object-image density for increasing λ. The probability $P_o(\Delta)$ plays a prominent roll in determining the autocovariance function of the scene-image model. Representative plots of $P_o(\Delta)$ are given in Figures 2-5. Figure 2 assumes that $p_w(W)$ equals $\delta(W-W_o)$. Notice in Figure 2 that $P_o(\Delta)$ is approximately triangular when λ is small. As λ increases, the probability that the same object image is visible for large $|\Delta|$ (i.e. $|\Delta|$ near W_o in Figure 2) decreases relative to that for smaller $|\Delta|$ due to occlusion. This relative change in probability causes $P_o(\Delta)$ of Figure 2 to depart from the triangular shape for large λ.

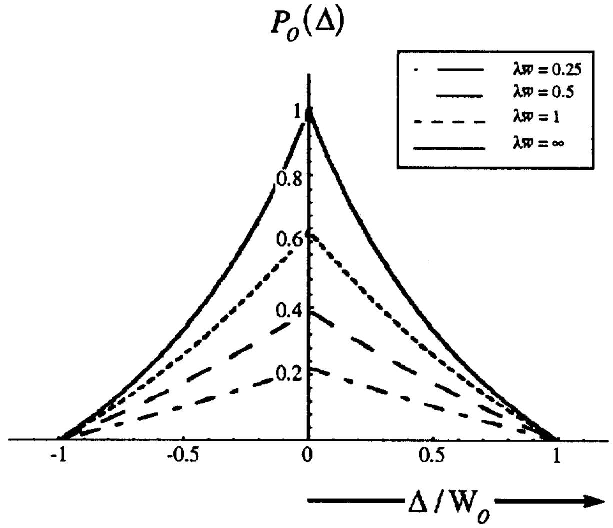

Figure 2. $P_o(\Delta)$: $p_w(W) = \delta(W - W_o)$

Figure 3 assumes that $p_w(W)$ is exponential: $p_w(W) = (1/\overline{w})exp\{-W/\overline{w}\}u(W)$ where $u(W)$ is the unit step function.

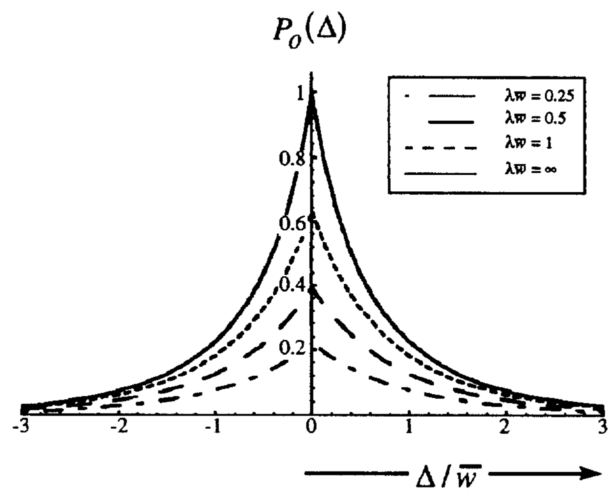

Figure 3. $P_o(\Delta)$: $p_w(W) = \frac{1}{\overline{w}}exp\{-W/\overline{w}\}u(x)$

Figures 4 and 5 assume that w is lognormal, that is, $ln(w)$ is normal with $E\{ln(w)\} = \mu$ and $\text{VAR}\{ln(w)\} = \sigma^2$. A lognormal distribution is of interest because it is claimed to describe the size distributions of various classes of physical objects [11]-[12]. Notice from Figure 5 that $P_o(\Delta)$ can have "tails" that extend over large Δ for lognormal w. These tails are related to significant correlation over large distances, Δ, in the autocovariance model of (27).

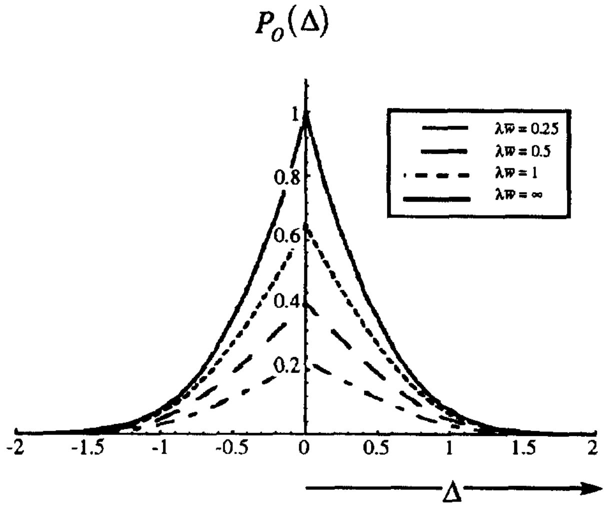

Figure 4. $P_o(\Delta)$**:** $p_w(W)$ **lognormal,** $\mu = 0$**,** $\sigma = 0.3$

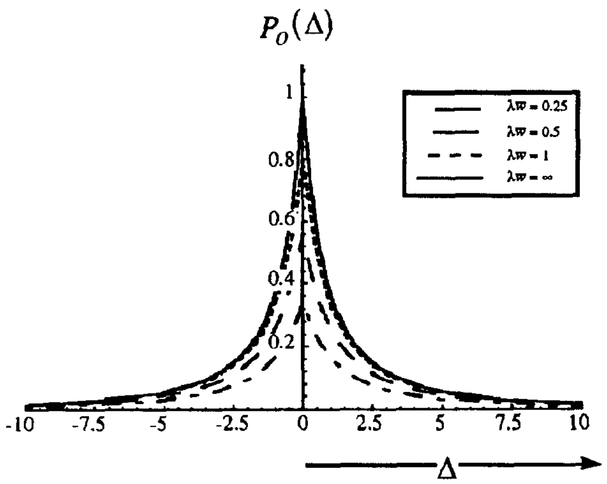

Figure 5. $P_o(\Delta)$**:** $p_w(W)$ **lognormal,** $\mu = 0$**,** $\sigma = 1$

For the general case of nonzero $\eta_{\mathbb{I}}$ and $\eta_b(x)$, each sample function $s(x,\zeta)$ of the random scene-image process $s(x)$ is the sum of $s'(x,\zeta)$ and $s''(x,\zeta)$ where $s''(x,\zeta) = \eta_{\mathbb{I}}$ if an object-image exists at x and $s''(x,\zeta) = \eta_b(x)$ otherwise. The autocorrelation function of $s(x)$ is therefore

$$R_s(x_1,x_2) = E\{s(x_1)s(x_2)\} =$$

$$E\{s(x_1)s(x_2)|O(x_1),O(x_2)\}\Pr\{O(x_1),O(x_2)\} +$$
$$E\{s(x_1)s(x_2)|O(x_1),B(x_2)\}\Pr\{O(x_1),B(x_2)\} +$$
$$E\{s(x_1)s(x_2)|B(x_1),O(x_2)\}\Pr\{B(x_1),O(x_2)\} +$$
$$E\{s(x_1)s(x_2)|B(x_1),B(x_2)\}\Pr\{B(x_1),B(x_2)\} =$$

$$\begin{aligned}&[K_{\mathbb{I}}(x_2-x_1)+\eta_{\mathbb{I}}^2]\Pr\{O(x_1),O(x_2)\} +\\ &\eta_{\mathbb{I}}\eta_b(x_2)\Pr\{O(x_1),B(x_2)\} +\\ &\eta_b(x_1)\eta_{\mathbb{I}}\Pr\{B(x_1),O(x_2)\} +\\ &R_b(x_1,x_2)\Pr\{B(x_1),B(x_2)\}\end{aligned} \quad (30)$$

The probabilities in (30) can be evaluated from previous results (see Appendix B). Evaluation of these probabilities yields:

$$R_s(x,x+\Delta) = K_s(x,x+\Delta) + \eta_s(x)\eta_s(x+\Delta) \quad (31)$$

where $\eta_s(x)$ is given by (16) and

$$\begin{aligned}K_s(x,x+\Delta) = {}& K_{\mathbb{I}}(\Delta)P_o(\Delta) + K_b(x,x+\Delta)P_b(\Delta)\\ &+ [\eta_{\mathbb{I}}-\eta_b(x)][\eta_{\mathbb{I}}-\eta_b(x+\Delta)][P_b(\Delta)-P_b(0)^2]\end{aligned} \quad (32)$$

In the limit $\lambda\to\infty$, the scene-image is composed exclusively of self-occluding object-images and the last two terms of (32) vanish. This yields

$$K_s(\Delta) = K_{\mathbb{I}}(\Delta)\left\{\frac{1-\psi(\Delta)}{1+\psi(\Delta)}\right\} \quad (33a)$$

If we assume further that $\mathbb{I}(x)$ is a random constant independent from one object image to the next, then (33a) reduces to

$$K_s(\Delta) = K_s(0)\left\{\frac{1-\psi(\Delta)}{1+\psi(\Delta)}\right\} \quad (33b)$$

where $K_s(0) = \mathrm{VAR}\{\mathbb{I}\}$. The assumption that $\mathbb{I}(x)$ is a random constant is analogous to that of Franks [9] and Schreiber [8].

3: Exponential Autocovariance

It is interesting to consider the implications of the symmetric-exponential autocovariance model $K_s(\Delta) = K_s(0)exp\{-\alpha \mid \Delta \mid\}$ on the object-image width pdf $p_w(W)$. If we equate (33b) to a symmetric-exponential, we obtain

$$\left\{\frac{1-\psi(\Delta)}{1+\psi(\Delta)}\right\} = e^{-\alpha|\Delta|} \quad (34)$$

which, by (25), implies that

$$P_w(W) = \left\{1 - \frac{4e^{-2W/\overline{w}}}{(1+e^{-2W/\overline{w}})^2}\right\}u(W) \qquad (35)$$

and

$$p_w(W) = \frac{8e^{-2W/\overline{w}}(1-e^{-2W/\overline{w}})}{\overline{w}(1+e^{-2W/\overline{w}})^3}\, u(W) \qquad (36)$$

where $\overline{w} = 2/\alpha$. A plot of $p_w(W)$ is shown in Figure 6.

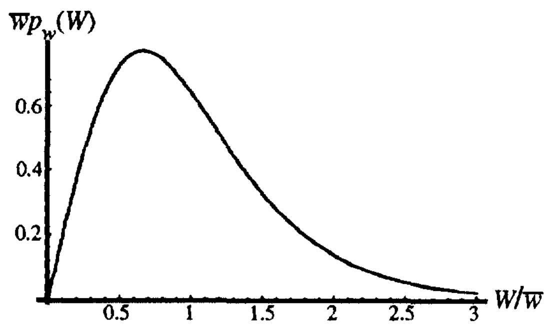

Figure 6. $p_w(W)$ **for symmetric-exponential** $K_s(\Delta)$

4: Conclusion

We have described an image model that captures the phenomenon of occlusion. We derived the relation between object-image width and scene-image autocovariance implied by the model. Since images are so varied, we are reluctant to accept the proposition that there is a "law of nature" which governs the distribution of object-image width in an *arbitrary* image. Yet, if we accept the image model of this paper, and if we also accept any autocorrelation model $K_s(\Delta)$, then we are implicitly accepting a constraint on object-image width distribution. Do width distributions exist which typically lead to more accurate image autocovariance models than the exponential autocovariance model? This intriguing question remains unanswered.

Appendix A

We show here the steps leading to (14) and (23). We start by writing ξ of (12) as

$$\xi = \int_0^\infty \left(1 - \tfrac{W}{2L} + \tfrac{W^2}{2L(2L+W)}\right) f_w(W)dW$$

$$= 1 - \tfrac{1}{2L}m_1(2L) + \tfrac{1}{2L}\int_0^\infty \tfrac{W^2}{2L+W} f_w(W)dW \qquad \text{A1}$$

where $m_1(2L)$ is the first moment of $f_w(W)$. The last term of A1 is bounded from below by zero. If we drop the W in the denominator of the integrand we obtain the upper bound $m_2(2L)/(2L)^2$ where $m_2(2L)$ is the second moment of $f_w(W)$. Therefore,

$$2\lambda L\ln(1 - \tfrac{1}{2L}m_1(2L)) < \ln\xi^{2\lambda L}$$

$$< 2\lambda L\ln(1 - \tfrac{1}{2L}m_1(2L) + \tfrac{1}{(2L)^2}m_2(2L)) \qquad \text{A2}$$

We use the inequalities

$$0 < m_1(2L) \le m_1(\infty) = \overline{w}, \qquad \text{A3}$$

$$0 < m_2(2L) \le m_2(\infty) = E\{w^2\}, \qquad \text{A4}$$

and

$$\epsilon - \epsilon^2 < \ln(1+\epsilon) < \epsilon \qquad \text{A5}$$

for $|\epsilon| < 0.5$ to bound the left and right-hand sides of A2. This yields, for sufficiently large L,

$$-\lambda m_1(2L) - \tfrac{\lambda}{2L}\overline{w}^2 < \ln\xi^{2\lambda L} < -\lambda m_1(2L) + \tfrac{\lambda}{2L}E\{w^2\} \qquad \text{A6}$$

It follows from A6 that $\xi^{2\lambda L} \to e^{-\lambda\overline{w}}$ as $L \to \infty$ which is the result used in (14).

Referring next to (22a) and (23) we have

$$\phi(\Delta) = \int_0^\infty \left\{\tfrac{2L-W}{2L+W}\left(1 + \tfrac{W}{2L-W}\mathbb{T}(\tfrac{\Delta}{W})\right)\right\} f_w(W)dw \qquad \text{A7}$$

We write the term in the brace of A7 as

$$1 - \tfrac{W}{L} + \tfrac{W}{2L}\mathbb{T}(\tfrac{\Delta}{W}) + \tfrac{W^2}{2L(2L+W)}\left(2 - \mathbb{T}(\tfrac{\Delta}{W})\right)$$

which yields

$$\phi(\Delta) = 1 - \tfrac{1}{L}m_1(2L) + \tfrac{1}{2L}\beta(\Delta, 2L) + \tfrac{1}{(2L)^2}\gamma(\Delta, 2L) \qquad \text{A8}$$

where

$$\beta(\Delta, 2L) \triangleq \int_0^\infty W\mathbb{T}(\tfrac{\Delta}{W}) f_w(W)dW \qquad \text{A9}$$

and

$$\gamma(\Delta, 2L) \triangleq \int_0^\infty \tfrac{2LW^2}{(2L+W)}\left(2 - \mathbb{T}(\tfrac{\Delta}{W})\right) f_w(W)dW \qquad \text{A10}$$

We find that $\gamma(\Delta, 2L) \le 2m_2(2L)$ by setting $\mathbb{T}$ to its minimum value, 0, and dropping the W from the denominator. By setting $\mathbb{T}$ to its maximum value, 1, we find that $\gamma(\Delta, 2L) \geq 0$. Similarly we find that $0 \leq \beta(\Delta, 2L) \leq \bar{\omega}$. Therefore,

$$2\lambda L \ln\left(1 - \frac{2m_1(2L) - \beta(\Delta,2L)}{2L}\right) < \ln\phi^{2\lambda L}(\Delta)$$

$$< 2\lambda L \ln\left(1 - \frac{2m_1(2L) - \beta(\Delta,2L)}{2L} + \frac{\gamma(\Delta,2L)}{4L^2}\right) \quad \text{A11}$$

By combining A11 with previous inequalities, we have, for sufficiently large L,

$$\lambda\left(-2m_1(2L) + \beta(\Delta, 2L)\right) - \frac{\lambda}{2L}(3\bar{\omega})^2 < \ln\phi^{2\lambda L}(\Delta)$$

$$< \lambda\left(-2m_1(2L) + \beta(\Delta, 2L)\right) + \frac{\lambda}{L}E\{w^2\} \quad \text{A12}$$

It is straightforward but lengthy to show that

$$\beta(\Delta, 2L) = m_1(2L) - m_1(2L)\psi(\Delta, 2L) \quad \text{A13}$$

where

$$\psi(\Delta, 2L) = \frac{1}{m_1(2L)} \int_0^{|\Delta|} [1 - F_\omega(W)]dW \quad \text{A14}$$

and $F_\omega(W)$ is the cdf associated with $f_\omega(W)$. $F_\omega(W) \to P_\omega(W)$ and $\psi(\Delta, 2L) \to \psi(\Delta)$ of (25) as $L \to \infty$. Therefore, as $L \to \infty$, $\phi^n(\Delta) \to e^{-\lambda\bar{\omega}[1+\psi(\Delta)]}$ which is a result appearing in (24).

To evaluate the limit of the factor $\frac{\phi(\Delta)-\theta(\Delta)}{1-\phi(\Delta)}$ in (23) we write

$$\frac{2L-W}{2L+W} = 1 - \frac{W}{L} + \frac{W^2}{L(2L+W)} \quad \text{A15}$$

so that $\theta(\Delta)$ of (22b) becomes

$$\theta(\Delta) = 1 - \frac{1}{L}m_1(2L) + \frac{1}{(2L)^2}\vartheta(\Delta, 2L) \quad \text{A16}$$

where

$$\vartheta(\Delta, 2L) \triangleq \int_0^\infty \frac{4LW^2}{(2L+W)} f_\omega(W)dW \quad \text{A17}$$

It is easy to show that $0 \leq \vartheta(\Delta, 2L) \leq 2E\{w^2\}$. It follows from A8 and A16 that

$$\frac{\phi(\Delta)-\theta(\Delta)}{1-\phi(\Delta)} = \frac{m_1(2L) - m_1(2L)\psi(\Delta,2L) + \frac{1}{2L}[\gamma(\Delta,2L) - \vartheta(\Delta,2L)]}{m_1(2L) + m_1(2L)\psi(\Delta,2L) - \frac{1}{2L}\gamma(\Delta,2L)} \quad \text{A18}$$

from which we have

$$\lim_{L\to\infty} \frac{\phi(\Delta)-\theta(\Delta)}{1-\phi(\Delta)} = \frac{1-\psi(\Delta)}{1+\psi(\Delta)} \quad \text{A19}$$

which is a result used in (24).

Appendix B

To evaluate the probabilities in (30) we first note that

$$\Pr\{B(x_2)|B(x_1)\} = \frac{\Pr\{B(x_1), B(x_2)\}}{\Pr\{B(x_1)\}} = \frac{P_b(\Delta)}{P_b(0)} = e^{-\lambda\bar{\omega}\psi(\Delta)} \quad \text{B1}$$

From B1, (29) and well-known rules of probability theory we get:

$$\Pr\{O(x_2)|B(x_1)\} = 1 - e^{-\lambda\bar{\omega}\psi(\Delta)} \quad \text{B2}$$

$$\Pr\{B(x_1)|O(x_2)\} = \frac{1-e^{-\lambda\bar{\omega}\psi(\Delta)}}{1-e^{-\lambda\bar{\omega}}} e^{-\lambda\bar{\omega}} \quad \text{B3}$$

$$\Pr\{O(x_1)|O(x_2)\} = \frac{1-2e^{-\lambda\bar{\omega}} + e^{-\lambda\bar{\omega}(1+\psi(\Delta))}}{1-e^{-\lambda\bar{\omega}}} \quad \text{B4}$$

$$\Pr\{O(x_1), O(x_2)\} = 1 - 2e^{-\lambda\bar{\omega}} + e^{-\lambda\bar{\omega}(1+\psi(\Delta))} \quad \text{B5}$$

$$\Pr\{O(x_2), B(x_1)\} = \left(1 - e^{-\lambda\bar{\omega}\psi(\Delta)}\right)e^{-\lambda\bar{\omega}} \quad \text{B6}$$

and

$$\Pr\{O(x_1), B(x_2)\} = \left(1 - e^{-\lambda\bar{\omega}\psi(\Delta)}\right)e^{-\lambda\bar{\omega}} \quad \text{B7}$$

Acknowledgments

It is a pleasure to thank Professors Thomas Huang, Nancy Hubing, William Schreiber and Seldon Trimble for helpful comments and encouragement. This work was supported by NSF Grant MIP-9223020.

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