Multiwindow Estimators of Correlation

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Abstract

Many algorithms for signal and array processing have embedded within them sample estimates of correlation. In this paper, we prove that the most general symmetric, quadratic, nonnegative-definite, modulation-invariant estimator of correlation is a multiwindow estimator. We establish that multiwindow estimators have the potential to reduce estimator mean-squared error by reducing variance at the expense of controllable bias. When multiwindow estimators are used to solve signal and array processing problems, they have the potential to improve and generalize many standard results.

1 Introduction

The estimation of correlations is a common problem in applications of signal processing and statistical inference. For example, power spectrum estimation, coherence analysis, interferometry, image registration, linear prediction, adaptive filtering, and beamforming use either explicit or implicit estimates of correlation. In this paper we develop a general theory for estimating correlations from experimental data. We prove a representation theorem for every quadratic estimator of a correlation matrix that is required to be Toeplitz symmetric, non-negative definite (nnd), and modulation-invariant. This is also a representation theorem for all symmetric, quadratic, nnd, modulation-invariant estimators of the correlation sequence. These estimators are shown to be Blackman-Tukey estimators [1] that are constructed from a few good windowings of experimental data and not from a single good windowing. When a multiwindow correlation estimator is used in a linear predictor or adaptive filter, a new multiadjustment filter results [7].

2 Spectrum Estimation

In reference [6], the multiwindow spectrum estimator of Thomson [9] was generalized to include all quadratic, modulation-invariant estimators. These estimators take the form

$$\begin{aligned}\hat{S}(e^{j\theta};\mathbf{x}) &= \left\|VD(e^{-j\theta})\mathbf{x}\right\|^2 \\ &= \mathbf{x}^{\mathrm{H}}D(e^{j\theta})V^{\mathrm{H}}VD(e^{-j\theta})\mathbf{x},\end{aligned}$$

where $\mathbf{x} = (x_0\, x_1 \cdots x_{N-1})^{\mathrm{T}} \in \mathbb{C}^{N\times 1}$ is time series or multisensor data, $D(e^{j\theta}) = \mathrm{diag}(\boldsymbol{\psi}(e^{j\theta}))$ is a modulation matrix, $\boldsymbol{\psi}(e^{j\theta}) = (1\; e^{j\theta} \cdots e^{j(N-1)\theta})^{\mathrm{T}}$ is a DTFT vector, and $V = (\mathbf{w}_i^{\mathrm{T}})$ is an $m \times N$ matrix whose rows $(\mathbf{w}_i)_1^m$ are the m windows of the estimator. This class of spectrum estimators includes the Schuster periodogram [8], the Daniell smoothed periodogram [3], the Grenander and Rosenblatt spectrograms [4], and the quadratic estimators of Clergeot [2] as special cases. The estimator $\hat{S}(e^{j\theta};\mathbf{x})$ is clearly non-negative and quadratic in the data. It is also modulation invariant:

$$\hat{S}(e^{j\theta}; D(e^{j\phi})\mathbf{x}) = \hat{S}(e^{j(\theta-\phi)};\mathbf{x}).$$

The spectrum estimator $\hat{S}(e^{j\theta};\mathbf{x})$ may be recast in a form which illustrates the role played by estimates of correlation matrices:

$$\begin{aligned}\hat{S}(e^{j\theta},\mathbf{x}) &= \mathrm{tr}\, VD(e^{-j\theta})\mathbf{x}\mathbf{x}^{\mathrm{H}}D(e^{j\theta})V^{\mathrm{H}} \\ &= \sum_{i=1}^{m}\mathbf{w}_i^{\mathrm{T}}D(e^{-j\theta})\mathbf{x}\mathbf{x}^{\mathrm{H}}D(e^{j\theta})\mathbf{w}_i^{*} \\ &= \boldsymbol{\psi}^{\mathrm{H}}(e^{j\theta})\sum_{i=1}^{m}W_i\mathbf{x}\mathbf{x}^{\mathrm{H}}W_i^{\mathrm{H}}\boldsymbol{\psi}(e^{j\theta}) \\ &= \boldsymbol{\psi}^{\mathrm{H}}(e^{j\theta})\hat{R}(\mathbf{x})\boldsymbol{\psi}(e^{j\theta})\end{aligned}$$

$$\hat{R}(\mathbf{x}) = \sum_{i=1}^{m}W_i\mathbf{x}(W_i\mathbf{x})^{\mathrm{H}}; \qquad W_i = \mathrm{diag}(\mathbf{w}_i).$$

This result suggests that $\hat{R}(\mathbf{x})$, a multiwindow estimator of the correlation matrix $R = E[\mathbf{x}\mathbf{x}^{\mathrm{H}}]$, might be

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fundamental. It is Hermitian symmetric, non-negative definite, and modulation-invariant:

1 $\hat{R}^{\mathrm{H}}(\mathbf{x}) = \hat{R}(\mathbf{x})$

2 $\mathbf{u}^{\mathrm{H}}\hat{R}\mathbf{u} = \sum_{i=1}^{m} |\mathbf{u}^{\mathrm{H}} W_i \mathbf{x}|^2 \geq 0$

3 $\hat{R}(D(e^{j\phi})\mathbf{x}) = \sum_{i=1}^{m} W_i D(e^{j\phi})\mathbf{x}(W_i D(e^{j\phi})\mathbf{x})^{\mathrm{H}} = D(e^{j\phi})\hat{R}(\mathbf{x})D(e^{-j\phi}).$

We argue that modulation invariance is an essential property of any estimator of correlation. Without it, spectrum estimators, linear prediction, and MUSIC produce anomalous results [5].

The matrix estimator $\hat{R}(\mathbf{x})$ uses multiple windowed copies of the original data $\mathbf{x}$, namely $W_i\mathbf{x}$, to estimate correlations:

$$\hat{R}(\mathbf{x}) = \sum_{i=1}^{m} \hat{R}_0(W_i\mathbf{x}) \tag{1}$$

$$R_0(\mathbf{x}) = \mathbf{x}\mathbf{x}^{\mathrm{H}}.$$

We think of this estimator as a rank-m multiwindow alternative to the usual rank-one sample covariance matrix estimator $\hat{R}_0(\mathbf{x}) = \mathbf{x}\mathbf{x}^{\mathrm{H}}$.

The spectrum estimator $\hat{S}(e^{j\theta};\mathbf{x})$ is invariant to Toeplitz averaging of $\hat{R}(\mathbf{x})$. That is, if $\hat{R}(\mathbf{x})$ is replaced by a Toeplitz matrix where each element on a diagonal is equal to the average of the elements on the corresponding diagonal of $\hat{R}(\mathbf{x})$, then $\hat{S}(e^{j\theta};\mathbf{x})$ remains unchanged. This means that the spectrum estimator $\hat{S}(e^{j\theta};\mathbf{x})$ may be written as a Fourier transform of an estimated correlation sequence $\hat{r}(t;\mathbf{x})$:

$$\hat{S}(e^{j\theta};\mathbf{x}) = \sum_{|t|<N} \hat{r}(t;\mathbf{x})e^{-jt\theta}$$

where

$$\begin{aligned}\hat{r}(t;\mathbf{x}) &= \mathrm{tr}\left[Z_t^{\mathrm{T}}\hat{R}(\mathbf{x})\right] \\ &= \mathrm{tr}\left[Z_t^{\mathrm{T}}\sum_{i=1}^{m} W_i\mathbf{x}(W_i\mathbf{x})^{\mathrm{H}}\right] \\ &= \mathbf{x}^{\mathrm{H}}Q_t\mathbf{x}\end{aligned}$$

$$Q_t = \sum_{i=1}^{m} W_i^{\mathrm{H}} Z_t^{\mathrm{T}} W_i = Q_{-t}^{\mathrm{H}}$$

$$Z_t = [\delta(t, i-j)]_{ij} = \begin{bmatrix} 0 & & \cdots & 0 \\ \vdots & & & \\ 1 & & & \vdots \\ & \ddots & & \\ 0 & & 1 \;\cdots & 0 \end{bmatrix} : \text{delay}.$$

In these formulas, Z_t is a delay matrix with the property that $\mathrm{tr}[Z_t^{\mathrm{T}}R]$ sums elements on the t-diagonal of R.

In words, the spectrum estimator $\hat{S}(e^{j\theta};\mathbf{x})$ is the Fourier transform of $\hat{r}(t;\mathbf{x})$, which is a quadratic, multiwindow estimator of the correlation sequence. It is a generalization of the Blackman and Tukey [1] spectrum estimators. This result suggests that $\hat{r}(t;\mathbf{x})$, a multiwindow estimator of the correlation sequence $r_t = E[x_{t+n}x_n^*]$, might be fundamental. It is Hermitian symmetric and modulation invariant:

(1) $\hat{r}^*(t;\mathbf{x}) = \hat{r}(-t;\mathbf{x})$

(2) $\hat{r}(t;D(e^{j\phi})\mathbf{x}) = e^{j\phi t}\hat{r}(t;\mathbf{x}).$

The estimator $\hat{r}(t;\mathbf{x})$ uses multiple-windowings of the data to estimate correlations:

$$\begin{aligned}\hat{r}(t;\mathbf{x}) &= \mathrm{tr}\left[Z_t^{\mathrm{T}}\sum_{i=1}^{m}(W_i\mathbf{x})(W_i\mathbf{x})^{\mathrm{H}}\right] \\ &= \mathbf{x}^{\mathrm{H}}\sum_{i=1}^{m} W_i^{\mathrm{H}} Z_t^{\mathrm{T}} W_i\mathbf{x} \\ &= \sum_{i=1}^{m}\hat{r}_0(t;W_i\mathbf{x})\end{aligned} \tag{2}$$

$$\hat{r}_0(t;\mathbf{x}) = \mathbf{x}^{\mathrm{H}} Z_t^{\mathrm{T}}\mathbf{x}.$$

We think of this estimator as a multiwindow or multirank alternative to the usual sample estimators $\hat{r}_0(t;\mathbf{x})$ and $\hat{r}_0(t;W\mathbf{x}) = \mathbf{x}^{\mathrm{H}}W^{\mathrm{H}}Z_t^{\mathrm{T}}W\mathbf{x}$, which are just sums of lagged inner products.

3 Representation Theorem

The results of the previous section *suggest* that the multiwindow estimators of the correlation matrix $R = E[\mathbf{x}\mathbf{x}^{\mathrm{H}}]$ and the correlation sequence $r(t) = E[x_{n+t}x_n^*]$ given in (1) and (2) are general. They further *suggest* that the following Toeplitz estimator of the correlation matrix is general:

$$\hat{R}(\mathbf{x}) = \sum_{|t|<N} \hat{r}(t;\mathbf{x})Z_t$$

$$\hat{r}(t;\mathbf{x}) = \mathbf{x}^{\mathrm{H}}\sum_{i=1}^{m} W_i^{\mathrm{H}} Z_t^{\mathrm{T}} W_i\mathbf{x}.$$

We will now state and prove a representation theorem which shows that this Toeplitz estimator and its associated Hermitian symmetric correlation sequence estimator are the most general quadratic, nnd, modulation-invariant estimators one can build.

Theorem: An estimator of the correlation coefficients $\{\hat{r}(t;\mathbf{x}),\ |t|<N\}$ is

1. non-negative definite;
2. Hermitian symmetric: $\hat{r}(-t;\mathbf{x}) = \hat{r}^*(t;\mathbf{x})$;
3. quadratic in the data (scale-invariant);
4. modulation-invariant: $\hat{r}(t; D(e^{j\theta})\mathbf{x}) = e^{jt\theta}\hat{r}(t;\mathbf{x})$

iff it may be computed as the multiwindow estimator

$$\hat{r}(t;\mathbf{x}) = \mathrm{tr}[Z_t^{\mathrm{T}}\hat{R}(\mathbf{x})];\ \hat{R}(\mathbf{x}) = \sum_{i=1}^{m}(W_i\mathbf{x})(W_i\mathbf{x})^{\mathrm{H}}$$

$$W_i = \mathrm{diag}(\mathbf{w}_i).$$

Proof: The sequence $\{\hat{r}(t;\mathbf{x})\}$ satisfies properties 2–4 iff [5]

$$\hat{r}(t;\mathbf{x}) = \mathbf{x}^{\mathrm{H}}Q_t\mathbf{x},$$

where

$$Q_t = Q_t^{\mathrm{H}};\quad D(e^{j\theta})Q_tD(e^{-j\theta}) = e^{-jt\theta}Q_t.$$

The sequence is nnd (property 1) iff the following induced Toeplitz correlation matrix and induced spectrum are, respectively, non-negative definite and non-negative:

$$\begin{aligned}\hat{R}(\mathbf{x}) &= \mathrm{Toeplitz}[\hat{r}(t;\mathbf{x})] \\ &= \int_{-\pi}^{\pi} \hat{S}(e^{j\theta};\mathbf{x})\psi(e^{j\theta})\psi^{\mathrm{H}}(e^{j\theta})\,\frac{d\theta}{2\pi} \geq 0\end{aligned}$$

$$\hat{S}(e^{j\theta};\mathbf{x}) = \sum_{|t|<N}\hat{r}(t;\mathbf{x})e^{-jt\theta} \geq 0.$$

Now substitute $\hat{r}(t;\mathbf{x}) = \mathbf{x}^{\mathrm{H}}Q_t\mathbf{x}$, with the quantity $D(e^{j\theta})Q_tD(e^{-j\theta}) = e^{-jt\theta}Q_t$, to write $\hat{S}(e^{j\theta};\mathbf{x})$ as

$$\begin{aligned}\hat{S}(e^{j\theta};\mathbf{x}) &= \sum_{|t|<N}\mathbf{x}^{\mathrm{H}}Q_t\mathbf{x}e^{-jt\theta} \\ &= \sum_{|t|<N}\mathbf{x}^{\mathrm{H}}D(e^{j\theta})Q_tD(e^{-j\theta})\mathbf{x} \\ &= \mathbf{x}^{\mathrm{H}}D(e^{j\theta})V^HVD(e^{-j\theta})\mathbf{x} \geq 0, \quad (3)\end{aligned}$$

where $Q = \Sigma Q_t = V^HV \geq 0$ must be nonnegative definite. But we have from (2) that the quadratic form of (3) may be written as

$$\hat{S}(e^{j\theta};\mathbf{x}) = \sum_{|t|<N}\hat{r}(t;\mathbf{x})e^{-jt\theta}$$

$$\hat{r}(t;\mathbf{x}) = \mathrm{tr}(Z_t^{\mathrm{T}}\hat{R}(\mathbf{x}));\quad \hat{R}(\mathbf{x}) = \sum_{i=1}^{m}(W_i\mathbf{x})(W_i\mathbf{x})^{\mathrm{H}}.$$

It follows that $\hat{r}(t;\mathbf{x}) = \mathbf{x}^{\mathrm{H}}Q_t\mathbf{x}$ may be written as $\mathrm{tr}(Z_t^{\mathrm{T}}\hat{R}(\mathbf{x}))$.

4 Mean-Squared Error Bounds

In reference [5], it is shown that

$$\int_{-\pi}^{\pi}\mathrm{mse}[\hat{S}(e^{j\theta};\mathbf{x})]\,\frac{d\theta}{2\pi} = \sum_{|t|<N}\mathrm{mse}[\hat{r}(t;\mathbf{x})] + \sum_{|t|\geq N} r_t^2.$$

That is, the integrated mean-squared error of the estimated spectrum is the accumulated mse for the N estimated correlation $\{\hat{r}(t;\mathbf{x});\ 0\leq t<N\}$ plus the square of the unmodeled correlation $\{r_t,\ t\geq N\}$. Using a mse bound from [6], it follows that

$$\begin{aligned}\sum_{|t|<N}\mathrm{mse}[\hat{r}(t;\mathbf{x})] &= \\ &= \int_{-\pi}^{\pi}\mathrm{mse}[\hat{S}(e^{j\theta};\mathbf{x})]\,\frac{d\theta}{2\pi} - \sum_{|t|\geq N} r_t^2 \\ &\geq \frac{1}{m+1}\sum_t r_t^2 - \sum_{|t|>N} r_t^2 \\ &= \frac{1}{m+1}\sum_{|t|<N} r_t^2 - \frac{m}{m+1}\sum_{|t|>N} r_t^2.\end{aligned}$$

That is, by choosing m windows correctly, one has the potential to decrease the accumulated mse by $\frac{1}{m+1}$.

5 Conclusions

The most general Hermitian symmetric, quadratic, nonnegative-definite, modulation-invariant estimator of correlation uses a multiplicity of very special windows. When multiwindow correlation estimates are used in spectrum analysis or wavenumber estimation (DOA estimation), then multiwindow spectrum estimators and multiaperture beamformers result. The multiaperture beamformers include subarray beamformers as special cases. When the results are applied to adaptive filtering, then multiadjustment adaptive filters result [7]. In such filters, each delay line's worth of data is windowed by several windows, and each such windowing is used to compute a minor weight adjustment. A multiplicity of such minor adjustments produces a major adjustment.

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