A Resolution Threshold for Multidimensional Parameter Estimates *

Michael P. Clark
Air Force Institute of Technology; AFIT/ENG
Wright-Patterson AFB, OH 45433-7765

Abstract

This paper presents a metric useful in analyzing the resolution of multidimensional parameter estimators which produce normal, unbiased estimates. The metric is based on the distance between ellipsoidal level curves of the individual estimates' densities. The resolution threshold is deemed to be the lowest SNR at which ellipsoids enclosing a given amount of probability mass have disjoint interiors. This paper develops an algorithm which computes the threshold by studying of the problem of finding the distance between two ellipsoids.

1 Introduction

Recent research into parametric techniques for multidimensional spectral estimation has been extensive [1, 2, 3, 4], etc. At high signal to noise ratios (SNRs) such techniques are preferable to classical nonparametric spectrum estimators in that the former algorithms offer the possibility of resolving parameters whose spacings are smaller than the Rayleigh limit. However, this is possible only when the variances of the parameter estimates are small in comparison to the parameter spacings. Since the variances of the estimates increase with decreasing SNR, one may define a resolution threshold noise level. This is the noise level at which the variances of the estimates are deemed to be commensurate with parameter spacings. For one-dimensional parameter estimation, definitions of resolvability closely related to the above were proposed by Oh and Kashyap [5] and Yau and Bresler [6].

2 A Resolution Metric

Consider a parameter estimation problem where, among the parameters of interest, are the vectors $\{\boldsymbol{\theta}_k\}_{k=1}^{p}$, where $\boldsymbol{\theta}_k \in \mathbb{R}^d$. Here each vector represents characteristics which distinguish a given feature of the data from other features. For instance, in the frequency-wavenumber estimation problem, $d = 4$ and each source is characterized by its frequency and three dimensional wavenumber vector. Next, consider an estimator which produces normally distributed estimates, $\{\hat{\boldsymbol{\theta}}_k\}_{k=1}^{p}$, of these parameter vectors. Specifically, suppose that each estimate is distributed as $\hat{\boldsymbol{\theta}}_k : \mathrm{N}[\boldsymbol{\theta}_k, C_k]$, where $C_k \in \mathbb{R}^{d\times d}$. That is, suppose that the density function for each $\hat{\boldsymbol{\theta}}_k$ is

$$
\begin{aligned}
f(\hat{\boldsymbol{\theta}}_k) &= (2\pi)^{-\frac{d}{2}} \det(C_k)^{-\frac{1}{2}} \\
&\cdot \exp\left(-\frac{1}{2}(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)^T C_k^{-1}(\hat{\boldsymbol{\theta}}_k - \boldsymbol{\theta}_k)\right),
\end{aligned}
\tag{1}
$$

where the covariance matrix is of the form $C_k \triangleq \sigma^2 R_k$, σ^2 represents the noise variance, and the estimates are unbiased. (The assumption of unbiasedness is justified for many techniques [1, 3, 7]).

Now, given the statistics of the estimates, one might desire to know when parameters which are closely spaced in d-space can be resolved. A possible answer to this question is obtained by superimposing plots of the density functions for the individual estimators and asking when the "peaks" can be distinguished. Of course, level curves of equation 2 are the boundaries of the ellipsoids

$$
\mathcal{E}_k \triangleq \{\boldsymbol{\theta} : (\boldsymbol{\theta} - \boldsymbol{\theta}_k)^T C_k^{-1}(\boldsymbol{\theta} - \boldsymbol{\theta}_k) \le r^2\}; \quad 1 \le k \le p \tag{2}
$$

where r^2 is some nonnegative constant. Among the ellipsoids associated with with each estimator, consider picking the one which encloses some given amount of probability mass, P. For P chosen large enough, one might use as a resolution metric the minimum of the distances between all pairs of ellipsoids. So long as this metric is greater than zero, all of the ellipsoids are disjoint, and the parameters can be resolved. Since the ellipsoids increase in size with σ^2, this metric suggests a threshold noise level for resolution. This is the maximum value of σ^2 where the ellipsoids have disjoint interiors. More formally, the problem of finding the resolution threshold is as follows:

*This work is supported in part by the Advanced Research Projects Agency under grant MDA 972-93-1-0015

Problem 1 *Consider p unbiased estimators $\{\hat{\theta}_k\}_{k=1}^p$ of some d-dimensional, real parameters $\{\theta_k\}_{k=1}^p$. Suppose that each $\hat{\theta}_k$ is normally distributed with positive definite covariance matrix $C_k = \sigma^2 R_k$. Suppose further that $\theta_k \neq \theta_l$ if $k \neq l$. Find the minimum value of σ^2 for which there exists a $\theta \in \mathbb{R}^d$, and integers $1 \leq k, l \leq p$ with $k \neq l$, such that*

$$(\theta - \theta_k)^T R_k^{-1} (\theta - \theta_k) \leq \sigma^2 r^2, \quad (3)$$

$$(\theta - \theta_l)^T R_l^{-1} (\theta - \theta_l) \leq \sigma^2 r^2. \quad (4)$$

Here r is chosen so that each ellipsoid in $\{\mathcal{E}_k\}_{k=1}^p$ encloses some given amount of probability mass, P. ([8] treats the problem of picking r^2 to give a desired value of P for the case of d-dimensional parameters.) Thus the problem is to find the minimum value of σ^2 such that at least two ellipsoids are tangent with disjoint interiors.

In order to identify the resolution threshold, this paper first treats the problem of finding the distance between two ellipsoids in d-space. Then, through the understanding gained from the distance problem, it develops a method for finding the value of σ^2 at which two ellipsoids become tangent. These topics are treated in the next two sections, respectively. Finally, the last two sections offer an example and the conclusion.

3 On the Distance between Ellipsoids

This section presents a general treatment of the problem of finding the distance between any two ellipsoids with disjoint interiors. Begin by defining the two ellipsoids

$$E_i \triangleq \{x : (x - \theta_i)^T R_i^{-1} (x - \theta_i) \leq \sigma^2 r^2\}, \quad (5)$$

$$E_j \triangleq \{y : (y - \theta_j)^T R_j^{-1} (y - \theta_j) \leq \sigma^2 r^2\}. \quad (6)$$

Define the distance between these ellipsoids to be

$$\mathrm{D}(E_i, E_j) = \min(\|x - y\|_A^2) : \quad x \in E_i, \ y \in E_j, \quad (7)$$

where for any $a \in \mathbb{R}^d$, $\|a\|_A \triangleq \|A^{-1}a\|_2$, and A is a nonsingular matrix. A weighted Euclidean norm is permitted because the problem of greatest interest in this paper is that of identifying the resolution threshold. At this threshold, the distance between the ellipsoids will be zero regardless of the weighting matrix chosen.

Next, assume E_i and E_j have disjoint interiors and that $\sigma^2 r^2 > 0$, $\sigma^2 r^2 > 0$. As R_i^{-1} and R_j^{-1} are positive definite by assumption, there exists a nonsingular matrix $T \in \mathbb{R}^{d \times d}$ such that

$$T^T R_i^{-1} T = D_x, \quad T^T R_j^{-1} T = I, \quad (8)$$

where $D_x \triangleq \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_d)$ is a positive definite matrix. T is said to simultaneously diagonalize R_i^{-1} and R_j^{-1} [9]. Choose $A = T$ and let

$$\tilde{x} \triangleq T^{-1}(x - \theta_i), \quad \tilde{y} \triangleq T^{-1}(y - \theta_i). \quad (9)$$

The problem then becomes

$$\min(\|\tilde{x} - \tilde{y}\|^2) : \quad \tilde{x} \in \tilde{E}_i, \ \tilde{y} \in \tilde{E}_j, \quad (10)$$

where

$$\tilde{y}_0 \triangleq T^{-1}(\theta_j - \theta_i), \quad (11)$$

$$\tilde{E}_i \triangleq \{\tilde{x} : \tilde{x}^T D_x \tilde{x} \leq \sigma^2 r^2\}, \quad (12)$$

$$\tilde{E}_j \triangleq \{\tilde{y} : (\tilde{y} - \tilde{y}_0)^T (\tilde{y} - \tilde{y}_0) \leq \sigma^2 r^2\}. \quad (13)$$

The transformed problem is that of finding the distance between an ellipsoid centered at the origin and a hypersphere. Of course, candidate solutions must be elements of the boundaries of $\tilde{E}_i$ and $\tilde{E}_j$. Thus the Lagrangian for the constrained minimization problem is

$$\mathcal{L}(\tilde{x}, \tilde{y}) = (\tilde{x} - \tilde{y})^T (\tilde{x} - \tilde{y}) - \lambda(\tilde{x}^T D_x \tilde{x} - \sigma^2 r^2) - \gamma[(\tilde{y} - \tilde{y}_0)^T (\tilde{y} - \tilde{y}_0) - \sigma^2 r^2]. \quad (14)$$

The first-order conditions necessary for $\tilde{x}_*$, $\tilde{y}_*$ to be a solution are

$$\tilde{x}_* - \tilde{y}_* - \lambda D_x \tilde{x}_* = 0, \quad (15)$$

$$\tilde{y}_* - \tilde{x}_* - \gamma(\tilde{y}_* - \tilde{y}_0) = 0, \quad (16)$$

$$\tilde{x}_*^T D_x \tilde{x}_* = \sigma^2 r^2, \quad (17)$$

$$(\tilde{y}_* - \tilde{y}_0)^T (\tilde{y}_* - \tilde{y}_0) = \sigma^2 r^2. \quad (18)$$

Note that equations 15 and 16 can be rewritten as

$$\tilde{x}_* = (I - \lambda D_x)^{-1} \tilde{y}_*, \quad (19)$$

$$\tilde{y}_* = \tilde{y}_0 + (1 - \gamma)^{-1} (\tilde{x}_* - \tilde{y}_0), \quad (20)$$

so long as the inverses exist. If $\gamma \neq 0$ these equations imply

$$\tilde{x}_* = (I - \alpha D_x)^{-1} \tilde{y}_0, \quad (21)$$

where

$$\alpha \triangleq -\frac{\lambda(1 - \gamma)}{\gamma}. \quad (22)$$

However, if $\gamma = 0$ then they are satisfied trivially by $\tilde{x}_* = \tilde{y}_*$ with $\lambda = 0$. The ellipsoids are tangent. In this case conditions 17 and 18 reduce to

$$\tilde{x}_*^T D_x \tilde{x}_* = \sigma^2 r^2, \quad (23)$$

$$(\tilde{x}_* - \tilde{y}_0)^T (\tilde{x}_* - \tilde{y}_0) = \sigma^2 r^2. \quad (24)$$

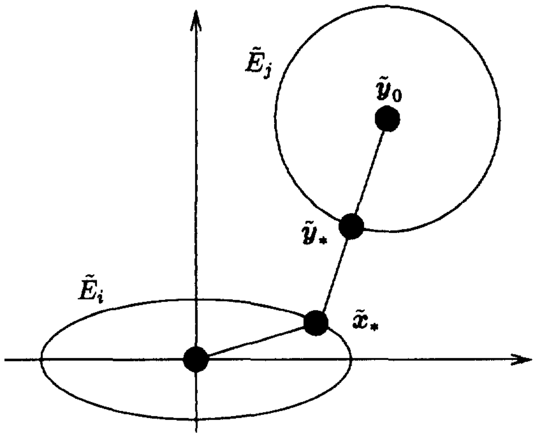

Figure 1: Geometry of the Distance Problem

So, when the ellipsoids are tangent, one must solve two simultaneous quadratic equations. Presented next, is an observation which leads one to suspect that the solution to these simultaneous quadratic equations is of the form of equation 21.

Consider the problem of finding the distance between the ellipsoid $\tilde{E}_i$ and the *center* of the hypersphere $\tilde{E}_j$. This distance is the solution to the minimization problem

$$\min_{\tilde{x}} \|\tilde{x} - \tilde{y}_0\|_2^2 : \quad \tilde{x} \in \tilde{E}_i \tag{25}$$

The Lagrangian for this problem is

$$\mathcal{L}(\tilde{x}) = (\tilde{x} - \tilde{y}_0)^T(\tilde{x} - \tilde{y}_0) - \alpha(\tilde{x}^T D_x \tilde{x} - \sigma^2 r^2). \tag{26}$$

The first-order conditions necessary for $\tilde{x}_*$ to be a solution are

$$\begin{aligned} \tilde{x}_* - \tilde{y}_0 - \alpha D_x \tilde{x}_* &= 0, & (27)\\ \tilde{x}_*^T D_x \tilde{x}_* &= \sigma^2 r^2. & (28) \end{aligned}$$

Notice that the first condition is equivalent to equation 21. Thus, so long as $\gamma \neq 0$ the problem of finding the distance between an ellipsoid and a hypersphere is essentially the problem of finding the distance from the ellipsoid to the center of the hypersphere. The solution to the latter problem yields the point $\tilde{x}_*$ on the ellipsoid which is closest to $\tilde{y}_0$. This solution is invariant to $\sigma^2 r^2$. This fact is illustrated by Figure 1 which shows a typical two-dimensional example. With $\tilde{x}_*$ determined, the distance between the ellipsoid and the hypersphere is easily found. This is seen from the figure and equation 20 which show that $\tilde{y}_*$ lies on the line connecting $\tilde{x}_*$ and $\tilde{y}_0$.

As noted before, when $\gamma = 0$, $\tilde{x}_* = \tilde{y}_*$, and the ellipsoids are tangent. In this case one might guess that the solution for $\tilde{x}_*$ is the same as when $\sigma^2 r^2$ is chosen small enough for the ellipsoids to be disjoint. This reasoning leads the following theorem.

Theorem 1 *Sufficient conditions for the optimization problem of equation 10 to have a unique, global minimum are*

$$\begin{aligned} \tilde{x}_* &= (I - \alpha D_x)^{-1}\tilde{y}_0, & (29)\\ \tilde{y}_* &= \tilde{y}_0 + \rho(\tilde{x}_* - \tilde{y}_0), & (30)\\ \sigma^2 r^2 &= \tilde{x}_*^T D_x \tilde{x}_*, & (31)\\ \alpha &< 0, & (32) \end{aligned}$$

where

$$\rho \triangleq \frac{\sigma r}{\|\tilde{x}_* - \tilde{y}_0\|_2}. \tag{33}$$

Proof of this theorem is provided in [8]. This reference also shows that a solution exists and gives an algorithm for its computation.

4 Identifying the Threshold

In order to find the threshold noise level, one must consider the distance between the ellipsoids associated with each pair of parameter estimates. Now, as σ^2 increases from zero, the distance between the ellipsoids decreases. The threshold noise level as the value of σ^2 at which the two ellipsoids are tangent but have disjoint interiors. Clearly, at threshold $\mathrm{D}(E_i, E_j) = 0$. The problem is to find the value of σ^2 which forces this condition while leaving the interiors disjoint.

When two ellipsoids with disjoint interiors are tangent, the point of tangency x_* satisfies $x_* = y_*$. Thus one might speculate that at the value of σ^2 where the two ellipsoids are tangent, the point of intersection should satisfy the conditions of Theorem 1 with $\tilde{x}_* = \tilde{y}_*$. This logic leads to the following theorem.

Theorem 2 *Sufficient conditions for the ellipsoids E_i and E_j to be tangent with disjoint interiors are*

$$\begin{aligned} \tilde{x}_* &= (I - \alpha D_x)^{-1}\tilde{y}_0, & (34)\\ \sigma^2 r^2 &= \tilde{x}_*^T D_x \tilde{x}_*, & (35)\\ \sigma^2 r^2 &= (\tilde{x}_* - \tilde{y}_0)^T(\tilde{x}_* - \tilde{y}_0), & (36)\\ \alpha &< 0. & (37) \end{aligned}$$

Proof Let $\tilde{x}_*$, σ^2 and α satisfy the conditions of the theorem. Then clearly $\tilde{x}_* \in \tilde{E}_i$ and $\tilde{x}_* \in \tilde{E}_j$. Suppose the interiors of $\tilde{E}_i$ and $\tilde{E}_j$ are not disjoint. (That is, that the solution is not unique.) Then there exists a nonzero vector $\epsilon \in \mathbb{R}^d$ such that $\tilde{x}_* + \epsilon$ is an element of both $\tilde{E}_i$ and $\tilde{E}_j$. One thus finds

$$\begin{aligned} \tilde{x}_* + \epsilon \in \tilde{E}_i \quad &\Leftrightarrow \quad (\tilde{x}_* + \epsilon)^T D_x (\tilde{x}_* + \epsilon) \le \sigma^2 r^2 \\ &\Leftrightarrow \quad \epsilon^T D_x \epsilon \le -2\epsilon^T D_x \tilde{x}_*, \end{aligned} \tag{38}$$

and

$$\begin{aligned} \tilde{x}_* + \epsilon \in \tilde{E}_j \quad &\Leftrightarrow \quad \|\tilde{x}_* + \epsilon - \tilde{y}_0\|^2 \le \sigma^2 r^2 \\ &\Leftrightarrow \quad \epsilon^T \epsilon \le -2\epsilon^T (\tilde{x}_* - \tilde{y}_0). \end{aligned} \tag{39}$$

Conditions 34, 37 and 38 give

$$\alpha \epsilon^T D_x \epsilon \ge -2\epsilon^T (\tilde{x}_* - \tilde{y}_0). \tag{40}$$

Together with condition 39 this implies

$$\alpha \epsilon^T D_x \epsilon \ge \epsilon^T \epsilon > 0. \tag{41}$$

Since D_x is positive definite and condition 37 dictates $\alpha < 0$, this is impossible. Thus the conditions are indeed sufficient for the two ellipsoids to be tangent with disjoint interiors. □

What is still to be shown is the existence of a solution satisfying the sufficient conditions. To show that one indeed exists, first substitute condition 34 into each of conditions 35 and 36. This yields

$$\frac{\tilde{y}_0^T D_x (I - \alpha D_x)^{-2} \tilde{y}_0}{r^2} = \sigma^2, \tag{42}$$

$$\frac{\alpha^2 \tilde{y}_0^T D_x^2 (I - \alpha D_x)^{-2} \tilde{y}_0}{r^2} = \sigma^2. \tag{43}$$

Together these equations imply

$$\frac{\tilde{y}_0^T D_x (I - \alpha D_x)^{-2} \tilde{y}_0}{\alpha^2 \tilde{y}_0^T D_x^2 (I - \alpha D_x)^{-2} \tilde{y}_0} = 1 \tag{44}$$

or, more compactly,

$$P(\alpha) \triangleq \frac{H(\alpha)}{G(\alpha)} = 1, \tag{45}$$

where

$$H(\alpha) \triangleq \tilde{y}_0^T D_x (I - \alpha D_x)^{-2} \tilde{y}_0 \tag{46}$$

$$G(\alpha) \triangleq \alpha^2 \tilde{y}_0^T D_x^2 (I - \alpha D_x)^{-2} \tilde{y}_0. \tag{47}$$

The derivative of $P(\alpha)$ with respect to α is

$$P'(\alpha) = \frac{H'(\alpha)G(\alpha) - H(\alpha)G'(\alpha)}{[G(\alpha)]^2}, \tag{48}$$

where

$$H'(\alpha) = 2\tilde{y}_0^T D_x^2 (I - \alpha D_x)^{-3} \tilde{y}_0 \tag{49}$$

$$G'(\alpha) = 2\alpha \tilde{y}_0^T D_x^2 (I - \alpha D_x)^{-3} \tilde{y}_0. \tag{50}$$

Now, when $\alpha < 0$ and $\theta_i \ne \theta_j$ one sees that

$$G(\alpha) > 0, \quad H(\alpha) > 0, \quad G'(\alpha) < 0, \quad H'(\alpha) > 0. \tag{51}$$

Thus from equation 48, $P'(\alpha) > 0$. So when $\alpha < 0$, $P(\alpha)$ is continuous and strictly increasing. Further, $\lim_{\alpha \to -\infty} P(\alpha) = 0$ and $\lim_{\alpha \to 0^-} P(\alpha) = \infty$. Thus by the intermediate value theorem there exists a unique solution to equation 45 satisfying $\alpha < 0$.

To find the solution, first notice that $G(\alpha)$ and $H(\alpha)$ can be expressed as

$$H(\alpha) \triangleq \frac{B(\alpha)}{A(\alpha)}, \quad G(\alpha) \triangleq \frac{\alpha^2 C(\alpha)}{A(\alpha)}, \tag{52}$$

where $A(\alpha)$ is a polynomial of degree $2d$, and $B(\alpha)$ and $C(\alpha)$ are a polynomials of degree $2d-2$ or less. Using this result the solution to equation 45 is seen to be the unique root of the degree $2d$ polynomial

$$\alpha^2 C(\alpha) - B(\alpha) \tag{53}$$

satisfying $\alpha < 0$. One can compute $B(\alpha)$ and $C(\alpha)$ using the following algorithm.

Algorithm 1
Given: $\tilde{y}_0$, D_x
Compute: $A(\alpha)$, $B(\alpha)$, $C(\alpha)$

$$\begin{aligned} &C_1(\alpha) = \beta_1^2 (\tilde{y}_0)_1^2 \\ &B_1(\alpha) = \beta_1 (\tilde{y}_0)_1^2 \\ &A_1(\alpha) = \beta_1^2 \alpha^2 - 2\beta_1 \alpha + 1 \\ &\textit{for } k = 2 \dots d \\ &\qquad C_k(\alpha) = \beta_k^2 \alpha^2 C_{k-1}(\alpha) - 2\beta_k \alpha C_{k-1}(\alpha) \\ &\qquad\qquad + C_{k-1}(\alpha) + \beta_k^2 (\tilde{y}_0)_k^2 A_{k-1}(\alpha) \\ &\qquad B_k(\alpha) = \beta_k^2 \alpha^2 B_{k-1}(\alpha) - 2\beta_k \alpha B_{k-1}(\alpha) \\ &\qquad\qquad + B_{k-1}(\alpha) + \beta_k (\tilde{y}_0)_k^2 A_{k-1}(\alpha) \\ &\qquad A_k = \beta_k^2 \alpha^2 A_{k-1}(\alpha) - 2\beta_k \alpha A_{k-1}(\alpha) + A_{k-1}(\alpha) \\ &A = A_d, \quad B = B_d, \quad C = C_d. \end{aligned}$$

Once $B(\alpha)$ and $C(\alpha)$ are computed, and the unique negative root of the polynomial of equation 53 is found, $\tilde{x}_*$ is computed using condition 34 and the threshold noise level, σ^2, is found by using either equation 42 or 43. Finally, problem 1 is solved by keeping the minimum threshold noise level obtained by using this procedure for all pairs of estimators in the set $\{\hat{\theta}_k\}_{k=1}^p$.

5 An Example

As an example consider a two-dimensional estimation scheme with $\theta_1 = [0 \ 10]^T$ and $\theta_2 = [-1 \ -1]^T$. Assume further that the covariance matrices for the

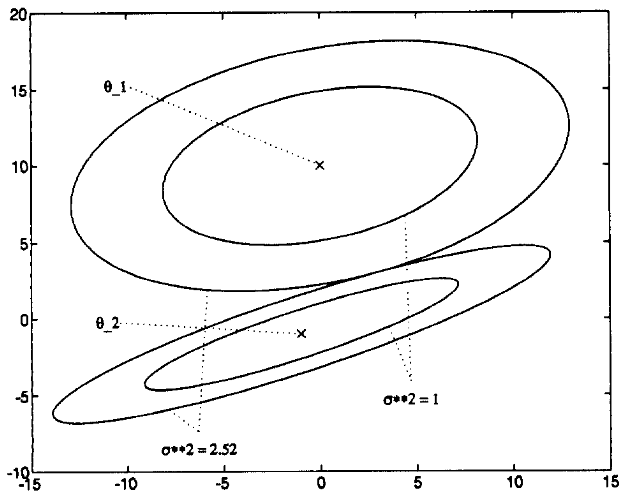

Figure 2: Ellipses of constant probability

two estimators are $C_1 = \sigma^2 R_1$ and $C_2 = \sigma^2 R_2$, respectively where

$$R_1 = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}. \tag{54}$$

The inner ellipses of figure 2 are the level curves enclosing 99% of the probability mass associated with the respective estimators. These are plotted for the case of $\sigma^2 = 1$. Obviously, at this noise level the parameters are resolvable. The threshold noise level is found as $\sigma_T^2 = 2.52$. The 99% ellipses for this value of σ^2 are also plotted in figure 2. Note they are tangent as required.

6 Conclusion

This paper developed a resolution metric for normally distributed vector parameter estimates. This metric depends on the distance between ellipsoids. Sufficient conditions were developed for finding such distances. These conditions guaranteed the existence of a unique solution. Based on the analysis of the distance problem, a simple method was developed for finding the noise level at which the ellipsoids become tangent. The solution to this problem was shown to exist so long as the centers of the ellipsoids were distinct.

References

[1] Y. Hua. Estimating two-dimensional frequencies by matrix enhancement and matrix pencil. *IEEE Transactions on Signal Processing*, SP-40(9):2267–2280, September 1992.

[2] A. Swindlehurst and T. Kailath. Algorithms for azimuth/elevation direction finding using regular array geometries. *IEEE Transactions on Aerospace and Electronic Systems*, AES-29(1), January 1993.

[3] M. P. Clark and L. L. Scharf. Two-dimensional modal analysis based on maximum likelihood. *IEEE Transactions on Signal Processing*, SP-42(6):1443–1461, June 1994.

[4] M. Zoltowski and C. Mathews. Closed-form 2D angle estimation with uniform circular arrays via phase mode excitation and ESPRIT. In *Conference Record of the Asilomar Conference on Signals, Systems and Computers*, pages 169–173, Pacific Grove, CA, 1993.

[5] S. Oh and R. Kashyap. A robust approach for high resolution frequency estimation. *IEEE Transactions on Signal Processing*, ASSP-39(3):627–643, March 1991.

[6] S. Yau and Y. Bresler. Worst case Cramer-Rao bounds for parametric estimation of superimposed signals with applications. *IEEE Transactions on Signal Processing*, SP-40(12):2973–2986, December 1992.

[7] P. Stoica and A. Nehorai. MUSIC, maximum likelihood, and the Cramer-Rao bound. *IEEE Transactions on Acoustics, Speech and Signal Processing*, ASSP-37(5):720–741, May 1989.

[8] M. P. Clark. On the resolvability of normally distributed vector parameter estimates. submitted to *IEEE Transactions on Signal Processing*, April 1994.

[9] Gene H. Golub and Charles F. Van Loan. *Matrix Computations*. Johns Hopkins, Baltimore, 1989.