Higher-Order Statistics: Laying a Myth to Rest

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Abstract

The property of the bispectrum that relates to random processes with symmetric probability density functions (PDFs) has often been interpreted to mean that the bispectrum of a stationary random process is zero if the marginal PDF is symmetric regardless of whether the random process is white or colored. It is shown that if the random process is not white, it can still have a non-zero bispectrum even if the marginal PDF is a symmetric function such as the Laplacian PDF or a nearly Gaussian function. Symmetry conditions expressed in terms of the marginal and joint PDFs that guarantee a zero bispectrum are postulated.

1 Introduction

Among the properties of third-order cumulant sequences and bispectra that come up for mention is one that says that they are identically zero for a random process whose probability density function (PDF) is symmetric. Examples of symmetric PDFs quoted in this context are the Laplacian and the uniform [1]. Usually the statement of the property is not accompanied by a qualification that the random process is white and the author has come across various instances of the property being interpreted as implying that the bispectrum of a discrete-time stationary random process with a symmetric marginal PDF is identically equal to zero regardless of whether or not the process is i.i.d. This is the myth being referred to in the title. This can perhaps be explained by the fact that in many applied fields, various problems are formulated in terms of the marginal PDF of a signal whether or not the signal is white. For example, a scalar PDF optimized waveform quantizer depends only on the marginal PDF [2]. A random signal, in this context, would be deemed to have a Laplacian distribution if its marginal PDF is Laplacian with no reference to whether it is i.i.d.

While the property (of the bispectrum being zero) holds for i.i.d random processes with symmetric marginal PDFs it does not necessarily hold when the process is colored. The first result here re-emphasizes this point by illustrating it with an example involving the Laplacian PDF. The second result is related to the bispectrum of a random process with a Gaussian marginal PDF. Although the property that higher order cumulants of a Gaussian random process are zero is clearly stated in terms of joint normality of samples of the random process [3] and although it is well known that if two random variables are Gaussian they are not necessarily jointly Gaussian, there have been instances of application where the mere fact that the signal under consideration had a marginal Gaussian PDF led to the abandoning of higher-order spectrum techniques. It is shown here, by means of an example, that a random process with a nearly Gaussian marginal PDF can have a bispectrum that is far from zero.

Finally, the paper provides, in the case where a random process is not i.i.d., a formulation of the property above in terms of the symmetries of the marginal PDF and joint PDFs of various orders.

2 Moving Average Processes with Zero Skewness

Consider the discrete time MA(q) random process $X(n)$ given by

$$X(n) = \sum_{i=0}^{q} a(i)W(n-i) \qquad (1)$$

where the $W(n)$s are i.i.d., $E\{W(n)\} = \overline{W}$ and $E\{[W(n) - \overline{W}]^3\} = \beta \neq 0$. The third order cumulant sequence $c(k,\ell)$ of $X(n)$ is given by [3]

$$c(k,\ell) = \beta \sum_{i=1}^{q} a(i)a(i+k)a(i+\ell). \qquad (2)$$

If the $a(i)$s are not all zero but the sum of their cubes is, then $c(0,0) = 0$ indicating that the skewness of the distribution of $X(n)$ is zero while the third order cumulant sequence of $X(n)$, and hence the bispectrum are not identically zero. Two interesting special cases are presented below.

2.1 Laplacian Random Process

If, in (2), $q = 1$, $a(0) = 1$ and $a(1) = -1$, we get

$$X(n) = W(n) - W(n-1). \tag{3}$$

As shown in [3], the bispectrum of this random process is not identically equal to zero even though the skewness of its distribution is.

That the PDF of $X(n)$ above is symmetric can also be seen from the fact that it is the convolution of the PDFs of $W(n)$ and $-W(n)$. Let $f_W(x)$ be the PDF of $W(n)$. Then, $f_X(x)$, the PDF of $X(n)$, is given by

$$f_X(x) = f_W(x) * f_W(-x) \tag{4}$$

which is equal to

$$\int_{-\infty}^{\infty} f_W(u) f_W(x+u)\, du. \tag{5}$$

The above is an autocorrelation function and, therefore, is symmetric about the origin.

An interesting illustration of this process is obtained when we make $W(n)$ of (3) exponentially distributed. The exponential PDF has been widely used in simulation studies of third order cumulant based algorithms for various purposes. Let

$$f_W(x) = \alpha \ \exp(-\alpha x) u(x); \ \alpha > 0, \tag{6}$$

where $u(x)$ is the unit step function. Substituting the above expression in (5) and evaluating the right hand side yields

$$f_X(x) = \frac{\alpha}{2} \exp(-\alpha |x|) \tag{7}$$

which is the Laplacian PDF. In many simulation studies of trispectrum based signal processing methods, the Laplacian distribution is used with the argument that its PDF being symmetric, the bispectrum of the corresponding random process is identically zero. As we have seen, the argument holds when the random process is i.i.d but not necessarily otherwise.

2.2 Nearly Gaussian Marginal PDF

We now show that even when the marginal PDF of the random process is symmetric and close to Gaussian, the bispectrum can be far from zero.

Let the exponentially distributed i.i.d. random process $W(n)$ from above pass through two systems in series with impulse responses $h_1(n)$ and $h_2(n)$ respectively where

$$h_1(n) = \delta(n) - \delta(n-1) \tag{8}$$

and

$$h_2(n) = \sum_{i=0}^{I-1} \frac{\delta(n-2i)}{\sqrt{I}}. \tag{9}$$

As shown above, the output of the first filter has a marginal PDF that is Laplacian. With $Y(n)$ denoting this output and $Z(n)$ denoting the output of the second filter, we have

$$Z(n) = \frac{1}{\sqrt{I}} \sum_{i=0}^{I-1} Y(n-2i). \tag{10}$$

For any n, $Z(n)$ is a sum of I independent identically distributed Laplacian random variables and has, by the central limit theorem, a marginal PDF that is close to Gaussian for large I. In fact, the marginal PDF approaches a Gaussian function rapidly as a function of I even though the normalized kurtosis decays slowly (as $1/I$). For example, for $I = 3$, the mean squared difference between the Gaussian function and the marginal PDF of $Z(n)$ is only 1.2% of the mean squared value of the corresponding Gaussian function. The variance of $Z(n)$ is simply the variance of the Laplacian random process going into the second filter.

However, for any finite I, the bispectrum of $Z(n)$ is not identically zero. The third moment sequence of $Z(n)$ is given by

$$c_Z(k,\ell) = \frac{\beta}{I^{3/2}} \sum_{i=0}^{2I-1} a(i)a(i+k)a(i+\ell) \tag{11}$$

where

$$a(i) = \begin{cases} (-1)^i & \text{for} \quad i = 0, 1, \ldots, 2I-1 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

and β is the centered third moment of $W(n)$ whose PDF is given in (6).

The bispectrum of $Z(n)$ is given by

$$B_Z(\omega_1, \omega_2) = \frac{\beta}{I^{3/2}} A(\omega_1) A(\omega_2) A^*(\omega_1 + \omega_2) \tag{13}$$

where

$$A(\omega) = \sum_{i=0}^{2I-1} a(i) \exp(-j\omega i) \tag{14}$$

$$= j \exp[-j(I - 1/2)\omega] \frac{\sin(I\omega)}{\cos(\omega/2)}. \tag{15}$$

Simplifying the above yields

$$B_Z(\omega_1,\omega_2) = -j\frac{\beta}{I^{3/2}}\frac{\sin(I\omega_1)}{\cos(\omega_1/2)}\frac{\sin(I\omega_2)}{\cos(\omega_2/2)} \times\frac{\sin[I(\omega_1+\omega_2)]}{\cos[(\omega_1+\omega_2)/2]}. \quad (16)$$

We will examine the behavior of $B_Z(\omega_1,\omega_2)$ for large values of I for the frequency pair $(\omega_1,\omega_2) = (\pi - \pi/2I, \pi/2I)$. We have [4]

$$|B_Z(\pi - \frac{\pi}{2I}, \frac{\pi}{2I})| = \frac{8|\beta|\sqrt{I}}{\pi} \quad (17)$$

for large values of I. Thus the bispectrum not only is not identically zero but actually increases in magnitude at various points as I increases. Furthermore, the bicoherence magnitude stays constant since we are dealing with a linear process.

3 Symmetry of Joint PDFs and Third-Order Statistics

Given that the symmetry of the marginal PDF by itself is not sufficient to conclude that the third-order statistics are zero, we will examine if the property can be stated in terms of the various joint PDFs associated with a random process. Since, unlike the marginal PDF, joint PDFs are multidimensional functions, the term symmetric should be qualified with an object of reference such as an axis. For example, if we examined the joint PDF of two successive values of the random process $X(n)$ in (3) with $W(n)$ having the PDF in (6) with $\alpha = 1$, we would find, with $\Theta = X(n)$ and $\Gamma = X(n-1)$ [4],

$$f_{\Theta\Gamma}(\theta,\gamma) = \begin{cases} \frac{1}{3}\exp[-(\theta+2\gamma)] & \gamma \geq 0 \text{ and } \theta+\gamma > 0 \\ \frac{1}{3}\exp[2\theta+\gamma] & \theta \leq 0 \text{ and } \theta+\gamma \leq 0 \\ \frac{1}{3}\exp[\gamma-\theta] & \text{otherwise.} \end{cases} \quad (18)$$

This function does not display symmetry with respect to any axis even though the marginal PDFs are Laplacian and it could be concluded that the non-zero third-order statistics arises from this asymmetry of the joint PDF.

However, it is also possible to have random processes with non-zero third-order statistics that have joint PDFs that exhibit symmetry about certain axes. For example, a pair of i.i.d. random variables X and Y with skewed marginal distributions have a joint pdf $f_{XY}(x,y)$ that is symmetric about the axis $y = x$.

It is clear then that any linkage between identically zero third-order statistics and PDF symmetry must involve the marginal as well as the joint PDFs of a random process. One choice that guarantees identically zero third-order statistics for a random process $X(n)$ is when all of the following are satisfied by the concerned joint PDFs:

1. For any n, $f_{X(n)}(x) = f_{X(n)}(-x)$,
2. For any $n \neq k$, $f_{X(n),X(k)}(x,y) = f_{X(n),X(k)}(x,-y)$ and $f_{X(n),X(k)}(x,y) = f_{X(n),X(k)}(-x,y)$, and
3. For distinct integers n, k and ℓ, $f_{X(n),X(k),X(\ell)}(x,y,z) = f_{X(n),X(k),X(\ell)}(x,y,-z)$, $f_{X(n),X(k),X(\ell)}(x,y,z) = f_{X(n),X(k),X(\ell)}(x,-y,z)$ and $f_{X(n),X(k),X(\ell)}(x,y,z) = f_{X(n),X(k),X(\ell)}(-x,y,z)$.

As can be seen, the above is a much more restrictive symmetry constraint than the one expressed in terms of the marginal PDF alone.

4 Conclusion

In those instances of application where the marginal PDF of a signal is known to be symmetric about the mean, third-order statistics need not be abandoned in favor of other even ordered statistics on the basis of this fact alone. If it is known that the signal is not white, it is worth investigating whether the bispectrum is non-zero using, for example, a test such as the one in [1] since it would be difficult to test for the symmetry conditions given above in terms of the joint PDFs.

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