Some Statistical Problems Associated with Signal Processing*

C. Radhakrishna Rao

Statistics Department
326 Classroom Building
Pennsylvania State University
University Park, PA 16802
e-mail: crr1@psuvm.psu.edu

Abstract

The object of this communication is to report on some recent results on statistical analysis and inference on signals in one and two dimensional superimposed exponential signal models, and to mention some outstanding problems in this area.

1 The One Dimensional Model

1.1 Estimation and asymptotic distribution

The one dimensional superimposed exponential model for signals is

$$y_t = \alpha_1 e^{i\omega_1 t} + \ldots + \alpha_p e^{i\omega_p t} + \varepsilon_t \qquad (1)$$
$$t = 1, \ldots, N$$

where $i = \sqrt{-1}, \alpha_s$ is the complex amplitude and ω_s is the frequency of the s-th signal, and

$$\begin{aligned} E(\varepsilon_t) = 0,\ E(\mathrm{Re}\,\varepsilon_t)^2 = E(\mathrm{Im}\,\varepsilon_t)^2 &= \sigma^2/2 \\ \mathrm{Cov}(\mathrm{Re}\,\varepsilon_t, \mathrm{Im}\,\varepsilon_t) &= 0. \end{aligned} \qquad (2)$$

For a given number p of signals, several methods of estimation are known like FBLP of Tufts and Kumaresan, EVLP (equivariation linear prediction) of Bai, Krishnaiah, Rao and Zhao, ML (maximum likelihood) and LS (least squares) as developed by Smyth, Osborne, Bresler, Macovski, Kundu, Stoica, Nehorai, Zhao and Rao. For exact references, the reader is referred to the books [2] and [4] and the papers [5] and [6].

Theorem 1. Let $\alpha_j \neq 0$, $\omega_j \neq \omega_k$ for $j \neq k$, $j, k = 1, \ldots, p$, $0 < \sigma^2 < \infty$, and $E(\mathrm{Re}\,\varepsilon_t)^4$ and $E(\mathrm{Im}\,\varepsilon_t)^4$ be finite. Then the least squares estimates $\hat{\underline{\omega}}$, $\hat{\underline{\alpha}}$ and $\hat{\sigma}^2$ of $\underline{\omega}, \underline{\alpha}$ and σ^2 based on N independent sample observations are strongly consistent and, the limiting distribution of

$$N^{3/2}(\hat{\underline{\omega}} - \underline{\omega}),\ N^{1/2}\,\mathrm{Re}\,(\hat{\underline{\alpha}} - \underline{\alpha}),\ N^{1/2}\,\mathrm{Im}\,(\hat{\underline{\alpha}} - \underline{\alpha}),\ N^{1/2}(\hat{\sigma}^2 - \sigma^2) \qquad (3)$$

is multivariate normal with the variance-covariance matrix

$$V = \sigma^2 \begin{pmatrix} 6B & 3A_1B & -3A_2B & 0 \\ 3A_1B & \frac{1}{2}I + \frac{3}{2}A_1^2B & -\frac{3}{2}A_2A_1B & 0 \\ -3A_2B & -\frac{3}{2}A_2A_1B & \frac{1}{2}I + \frac{3}{2}A_2^2B & 0 \\ 0 & 0 & 0 & \sigma^2 \end{pmatrix}$$

where

$$A_1 = \mathrm{Im}\ A\ ,\ A_2 = \mathrm{Re}\,A,\ B = (A^*A)^{-1}$$
$$A = \ \mathrm{diag}\,(\alpha_1, \ldots, \alpha_p). \qquad (4)$$

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Note that the result of Theorem 1 is true without the assumption of normality of the error term ε_t in the model (1), as shown in [3]. If ε_t has a complex normal distribution, the condition on the fourth moments of ε_t is not necessary, and further it can be claimed that the matrix V in (4) is the lower (CR) bound to the asymptotic covariance matrix of the estimates as shown in [5].

1.2 Joint estimation of the number of signals and signal parameters

Let p, the true number of signals in the model (1), be not greater than a chosen number P, and compute

$$S_p = \min \frac{1}{N-p} \sum_{n=p+1}^{N} |\sum_{m=0}^{p} b_m^{(p)} \overline{y(n-m)}|^2$$

subject to $|b_0^{(p)}|^2 + \ldots + |b_p^{(p)}|^2 = 1$. Further let $R_p = S_p + p\, C_N$, where C_N is chosen such that

(i) $\lim C_N \to 0$ as $N \to \infty$,

(ii) $\lim \sqrt{N}\ C_N / \sqrt{\log\log N} = \infty$ as $N \to \infty$.

Then estimate p by $\hat{p}$ where

$$R_{\hat{p}} = \min\{R_0, T_1, \ldots, R_P\}.$$

Let $\hat{b} = (\hat{b}_0, \ldots, \hat{b}_{\hat{p}})$ be such that

$$S_{\hat{p}} = \frac{1}{N-\hat{p}} \sum_{n=\hat{p}+1}^{N} |\sum_{m=0}^{\hat{p}} \hat{b}_m \overline{y(n-m)}|^2$$

and $\rho_j e^{i\hat{\omega}_j}$, $j = 1, \ldots, \hat{p}$ be the roots of

$$\hat{H}(z) = \sum_{j=0}^{\hat{p}} \hat{b}_j\, x^j.$$

Then $\hat{\omega}_j$ is an estimate of ω_j, $j = 1, \ldots, \hat{p}$.

Theorem 2

If $E\{e^{h|\varepsilon(t)|^2}\} < \infty$ for some $h > 0$, then:

(i) Prob.$(\hat{p} \neq p) = O(e^{-\delta_1 N C_N^2})$ for some $\delta_1 > 0$.

(ii) Prob.$\{|\hat{\underline{b}} - \underline{b_p}| > \varepsilon\} = O(e^{-\delta_2 N C_N^2})$ for some $\delta_2 > 0$.

(iii) Prob.$\{\max|\hat{\omega}_m - \omega_m| \geq \varepsilon$ over $1 \leq m \leq p \wedge \hat{p}\}$ $= O(e^{-\delta_3 N C_N^2})$ for some $\delta_3 > 0$.

Note that in Theorem 2, the vectors $\hat{\underline{b}}$ and $\hat{\underline{b_p}}$ may not be of the same dimensions. The usual norm is used if they have the same dimension and if not, the norm is defined to be ∞. It would be of great interest to make further investigations on the asymptotic distribution of $\hat{\underline{b}}$ and study the consequences of prior estimation of p, the number of signals.

2 The two dimensional model

Let us consider the two dimensional superimposed exponential signals model

$$\begin{aligned} y(s,t) &= \sum_{j=1}^{p}\sum_{k=1}^{q} \gamma_{jk} e^{i(s\mu_j + t\nu_k)} + \varepsilon(s,t) \\ s &= 0, \ldots, m-1,\ t = 0, \ldots, n-1 \end{aligned} \quad (5)$$

and the maximum likelihood estimates of the parameters $\gamma_{jk}, \mu_j, \nu_k$ for all j and k and σ^2 (as defined in condition (1) below). We prove the following theorem.

Theorem 3. Let:

1) $\varepsilon(s,t) \sim CN(0, \sigma^2)$, $0 < \sigma^2 < \infty$

2) $\mu, \ldots, \mu_p$ be different and so also $\nu_1, \ldots, \nu_q$

3) $\sum_k \gamma_{jk}^2 > 0$, $\sum_j \gamma_{jk}^2 > 0$

4) There exist positive constants C_1, C_2, α_1 and α_2 such that

$$C_1 n^{\alpha_1} \leq m \leq C_2 n^{\alpha_2}.$$

Under these four assumptions, the asymptotic distribution of the ML estimates $\hat{\mu}, \hat{\nu}, \hat{\gamma}$ and $\hat{\sigma}^2$ suitably normalized

$$m^{3/2} n^{1/2} (\hat{\underline{\mu}} - \underline{\mu}), m^{1/2} n^{3/2} (\hat{\underline{\nu}} - \underline{\nu}),$$

$$\sqrt{mn}\,\mathrm{Re}\,(\hat{\underline{\gamma}} - \underline{\gamma}), \sqrt{mn}\,\mathrm{Im}\,(\hat{\underline{\gamma}} - \underline{\gamma})$$

$$\sqrt{mn}\,(\hat{\sigma}^2 - \sigma^2)$$

is multivariate normal with the optimal variance-covariance matrix $\sigma^2\Sigma = \sigma^2(\Sigma_{ij})$, $i, j = 1, \ldots, 5$..

The Σ_{ij} are defined as follows:

$$\begin{aligned}
\Sigma_{11} &= 6\Gamma_\Delta^{-2}, \Sigma_{13} = \Sigma'_{31} = 3\mathrm{Im}(\Gamma_\Delta^{-2}G'_\Delta) \\
\Sigma_{14} &= \Sigma'_{41} = -3\mathrm{Re}(\Gamma_\Delta^{-2}G'_\Delta), \Sigma_{22} = 6\Gamma_\delta^{-2} \\
\Sigma_{23} &= \Sigma'_{32} = 3\mathrm{Im}(\Gamma_\delta^{-2}G'_\delta) \\
\Sigma_{24} &= \Sigma'_{42} = -3\mathrm{Re}(\Gamma_\delta^{-2}G'_\delta) \\
\Sigma_{33} &= \frac{3}{2}(\mathrm{Im}(G_\Delta)\Gamma_\Delta^{-2}\mathrm{Im}(G'_\Delta) \\
&+ \mathrm{Im}(G_\delta)\Gamma_\delta^{-2}\mathrm{Im}(G'_\delta)) + \frac{1}{2}I_{pq} \\
\Sigma_{34} &= \Sigma'_{43} = -\frac{3}{2}(\mathrm{Im}(G_\Delta)\Gamma_\Delta^{-2}\mathrm{Re}(G'_\Delta) \\
&+ \mathrm{Im}(G_\delta)\Gamma_\delta^{-2}\mathrm{Re}(G'_\delta)) \\
\Sigma_{44} &= \frac{3}{2}(\mathrm{Re}(G_\Delta)\Gamma_\Delta^{-2}\mathrm{Re}(G'_\Delta) \\
&+ \mathrm{Re}(G_\delta)\Gamma_\delta^{-2}\mathrm{Re}(G'_\delta)) + \frac{1}{2}I_{pq} \\
\Sigma_{55} &= \sigma^2
\end{aligned}$$

and the rest of the Σ_{ab} are zero. In the above formula

$$\begin{aligned}
\Gamma_\Delta &= \mathrm{diag}(\|\gamma_1\|, \ldots, \|\gamma_p\|) : p \times p \quad \text{matrix} \\
\Gamma_\delta &= \mathrm{diag}(\|_1\gamma\|, \ldots, \|_q\gamma\|) : q \times q \quad \text{matrix} \\
G_\Delta &= \mathrm{diag}(\gamma_1, \ldots, \gamma_p) : pq \times p \quad \text{matrix} \\
G_\delta &= \mathrm{diag}(_1\gamma, \ldots, _q\gamma) : pq \times q \quad \text{matrix} \\
\gamma_j &= (\gamma_{j1}, \ldots, \gamma_{jq})', _j\gamma = (\gamma_{1j}, \ldots, \gamma_{pj})'
\end{aligned}$$

The same result holds for any distribution of $\varepsilon(s,t)$, with finite fourth moments. For details the reader is referred to [6] and [3].

Some problems of interest in this area are the joint estimation of p, q and the signal parameters as in the one dimensional case. Up to now no stable algorithm has been found for computing the ML estimates, although some methods exist for obtaining estimates which have the same asymptotic properties as the ML estimates (see for instance the paper [1]). Further research in this direction will be of interest.

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