Experimental Validation of the Kronecker Product
Godard Blind Adaptive Algorithms

Hamadi Jamali *, Sally L. Wood * and Roberto Cristi **

* Electrical Engineering Department
Santa Clara University, Santa Clara, CA 95053
** Electrical and Computer Engineering Dept.
Naval Postgraduate School, Monterey, CA 93943

Abstract

To overcome the analytical challenge the non convex Godard criterion presents, the kronecker product operator and higher order statistical moments have been proposed as a natural framework to obtain an exact and more tractable mathematical description of the behavior of the algorithm [2]-[4]. This paper validates these findings by working out several examples in both notations. Particular attention is given to situations of standard Constant Modulus Adaptive (CMA) algorithm misbehavior.

1. Overview of the Godard Algorithm

The Godard, or the Constant Modulus Adaptive algorithm as is also known, is a mechanism for adjusting the coefficients of an equalizer when the transmitted sequence is not observable. Unlike “template” matching methods, where the output is forced to follow some reference signal, this update rule proposes a non convex function based on the output measurements alone as its criterion of goodness. A comprehensive account of available literature as well as some open questions can be found in [1]. A rigorous analytical description of the behavior of the algorithm via classical techniques of linear prediction theory has, however, proven to be very difficult.

As a step toward overcoming this difficulty, [2] offers the kronecker product operator, combined with higher order statistical moments as a natural framework for better visualization of the cost function of the Godard equalizer. This change of notation is shown to yield itself well to classical methods of analysis and thus allowing for a deeper theoretical understanding of the behavior of the algorithm. In [3], this new representation is used to put forward an exact mathematical analysis of the behavior of the 2-2 Constant Modulus Adaptive algorithm. This analysis includes the derivation of higher order moments based “Yule-Walker” equations for the Godard equalizer, the derivation of closed form expressions for both its desired and undesired minima, and a description of the geometry of the stationary surfaces of its performance measure. The relationship of the desired minima to the true inverse channel in the case of both minimum and non minimum phase systems is also investigated in [3]. In [4], two variations of the 2-2 CMA rule in terms of the kronecker product of the weight vector are introduced. Unlike the original update law, these algorithms are globally convergent and do not require elaborate initialization schemes. Practical implementations for both minimum and non minimum phase systems are also discussed.

This paper attempts to verify the mathematics of references [2]-[4] on several examples representing different channel structures.
These examples are worked out both analytically and through computer simulations. Results from both the algorithms proposed in [4] as well as from the standard CMA rule are given. Issues of particular interest in this study are (i) the location of the desired equalizers, (ii) the existence of undesired minima, (iii) the relationship of the optimum equalizer to the true inverse channel under imperfect conditions, and (iv) the global convergence of the kronecker product version of the CMA (KPCMA).

2. Summary of Relevant Equations

This section lists some of the results from both the standard CMA and the KPCMA CMA algorithms that are of interest to this study. Due to space constraints, these formulas are given without any proof. The interested reader is referred to references [2]-[4] for the details of these derivations. A block diagram representation of the Godard algorithm when used for channel equalization is shown in Figure 1. Typically, the output $y_t$ of the equalizer is described by

$$y_t = W^*X_t$$

(1)

where $W$ is the weight vector, $X_t$ is the taped delay line data vector, and the symbol $*$ stands for the transpose combined with the complex conjugation operations.

The update law corresponding to the 2-2 CMA is defined as

$$W_t = W_{t-1} - \mu \varepsilon_t y_t X_t$$

(3)

where $\mu$ is the adaptation gain and

$$\varepsilon_t = |y_t|^2 - 1.$$

The update law for the 2-2 CMA performance becomes

$$J = \frac{1}{4} \theta^* R_{\varphi\varphi} \theta - \frac{1}{2} \theta^* P_{\varphi} + \frac{1}{4}$$

(4)

where $R_{\varphi\varphi} = E[\varphi_t \varphi^*_t]$ is formed of fourth order moment elements and $P_{\varphi} = E[\varphi_t]$ contains second order moment elements.

The undesired minima obey the condition

$$Q(W)W = 0$$

(6)

where the elements of the matrix $Q(W)$ are of the form

$$q_{ij} = W^* R_{il+j} W - p_{il+j}$$

(7)

where $R_{il+j}$ is a matrix formed from the $il+j$ row of $R_{\varphi\varphi}$.

A steepest descent type algorithm can be used to update the kronecker product vector as

$$\theta_t = \theta_{t-1} - \mu \delta \theta$$

(8)

where $\delta \theta$ is the gradient vector of $J$ with respect to $\theta$. 


Similarly, a stochastic gradient based algorithm in terms of the kronecker product is given by
\[ \theta_t = \theta_{t-1} - \mu \mathbf{e}_t \mathbf{\varphi}_t \]  
where \( \mu \) is chosen as
\[ 0 \leq \mu \leq \frac{2}{\lambda_{\text{max}}} \]
with \( \lambda_{\text{max}} \) being the largest eigenvalue of the matrix \( R_{\varphi \varphi} \).

3. Experimental Results

A total of six different examples are worked out in this section.

3.1. Example 1

The channel is represented by
\[ x_t + 0.6 x_{t-1} = s_t \]
The equalizer is chosen as \( y_t = w_0 x_t + w_1 x_{t-1} \)
Calculating the quantities \( R_{\varphi \varphi} \) and \( P_{\varphi} \), the desired kronecker product optimum is
\[ (W \otimes W) = \begin{bmatrix} 1 \\ 0.6 \\ 0.36 \end{bmatrix} \]
In this case, the obtained optimum is a kronecker product form and corresponds to the true parameters of the inverse channel. The undesired stationary points are at
\[ \pm \begin{bmatrix} 0 \\ 0.3680 \\ 0.6562 \end{bmatrix}, \pm \begin{bmatrix} 0 \\ 0.3680 \\ -0.2109 \end{bmatrix}, \pm \begin{bmatrix} 0 \\ 0.5575 \end{bmatrix}. \]
A closer look at the hessian matrix reveals that only the last one of these vectors is a minimum point. The first vector corresponds to a maximum point and the rest to saddle points. These remarks are illustrated in figures 2 to 4. Notice that the standard CMA may converge to undesired weights, the output of the equalizer is not a constant modulus.

3.2. Example 2

The channel is assumed to be an AR(2) with poles at -0.6 and -0.3. The equalizer is an MA(2). In this case also, the computed desired kronecker optimum has the kronecker form and corresponds to the \( \pm \) the true parameters of the inverse channel. Figure 5 shows that the CMA can misbehave, while figure 6 illustrates the global convergence of the KPCMA.

3.3. Example 3

This is the same structure as in example 2, except that the channel now has double pole at 0.9. This example exhibits the same behavior as the previous one, except that \( \mu \) in this case must be of the order of \( 10^{-8} \). A \( \mu \) of 0.01 is adequate for the first two examples.

3.4. Example 4

The channel is the same as the one in example 1. However, the equalizer is now an MA(2). As predicted in [3], this is a degenerate case. Notice here that while these examples so far may not reflect practical cases, they serve to illustrate the predictions of references [2]-[4] for idealized situations.

3.5. Example 5

Example 1 is reconsidered here when an additive noise is present at the output of the channel. The signal to noise ratio is assumed to be 10. In this case, the computed desired kronecker product vector is
\[ (W \otimes W) = \begin{bmatrix} 0.95 \\ 0.5663 \\ 0.3415 \end{bmatrix} \]
which, as expected, is different (but close) from the true inverse channel and does not
have the kronecker product property. The global convergence of the KPCMA to these weights is shown in figure 7.

3.6. Example 6

The channel is now modeled as an MA(1) with zero at 0.6. The true equalizer is

\[ y_t = \sum_{i=0}^{\infty} (-0.6)^i x_{t-i} \]

However, if an MA(1) equalizer only is assumed, then, the computed desired kronecker product vector in this is

\[ (W \otimes W) = \begin{bmatrix} 0.5180 \\ -0.2168 \\ 0.2867 \end{bmatrix} \]

which of course is different from the true inverse channel and does not have the correct form. This example, however, illustrate that, the existence of local minima is not restricted to auto regressive channels as reported in figure 8. Figure 9 shows that the KPCMA does not suffer from this problem.

When the length of the equalizer is increased to eight taps, then the computed desired kronecker product vector is very close to the first eight coefficients of the true equalizer.

4. Concluding Remarks

This experimental study has illustrated that the analysis of [2]-[4] accurately predicts the behavior of the Godard algorithm under various conditions. However, due to finite length approximation and measurements noise, the desired kronecker product vector does not in general have the correct form. This makes the extraction of the equalizer weights a difficult task. Means for handling this difficulty are investigated in [4].

4. References


