An Open-Loop Feedback Control Approach to Point-to-Point Control of Linear Continuous-Time Time-Invariant Dynamic Systems

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Abstract

In this paper, a combination of the classical frequency-domain and modern state space approaches are employed to obtain an open-loop feedback control for a special class of control systems known as point-to-point (or terminal) control systems. The developed method is utilized in the derivation of the open-loop feedback control.

Introduction:

Many researchers have studied a special class of point-to-point control problems, namely, the two-point-boundary-value problems [1, 2, 3]. In two-point-boundary-value problems, certain optimality conditions are required. In this paper, without requiring any optimality conditions, the point-to-point control problems are studied.

In general, a point-to-point control problem can be described mathematically as follows: given a linear continuous-time time-invariant dynamic system modeled by a set of differential equations of the form

\[ x(t) = Ax(t) + Bu(t) \]  

where constant coefficients \( A \) and \( B \) are \( nxn \) and \( nx1 \), and state \( x(t) \) and input \( u(t) \) are \( nx1 \) and \( 1x1 \) matrices respectively, determine the control input \( u(t) \) so that the state \( x(t) \) satisfies the boundary conditions \( x(t_0) = x_0 \) and \( x(t_f) = x_f \).

Theory:

It can be proved that [4, 5]

\[ x(t) = \text{EXP}(A(t - t_0))x(t_0) + \int_{t_0}^{t} \text{EXP}(A(t - \xi))Bu(\xi)d\xi \]  

and assuming full controllability, a unique transformation matrix \( T \) can be found to transform Eqn. (1) into Jordan canonical form

\[ \ddot{x}(t) = \bar{A}\dot{x}(t) + \bar{B}u(t) \]  

where

\[ \bar{A} = T^{-1}AT = \begin{bmatrix} A_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & A_k \end{bmatrix} \]

and

\[ \bar{B} = T^{-1}B = \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix} \]

where

\[ \lambda_i \]

are the eigenvalues of \( \bar{A} \) and

\[ A_i = \begin{bmatrix} 1 & \ldots & 0 \\ \lambda_i & \ddots & \vdots \\ 0 & \ldots & 1 \end{bmatrix} \]
Denoting the Finite Laplace Transform of \( u(t) \) as \( U(s, t_0, t_1) \), since
\[
\int_{t_0}^{t_1} \exp(-s \tau) u(t) \, dt = \frac{1}{s} \frac{d}{ds} U(s, t_0, t_1) \bigg|_{s = \lambda_i}
\]
Eqn. (5) can be written as
\[
\exp[-A_{1t}] \bar{x}(t) = \exp[-A_{10}] \bar{x}(t_0) + \left( U(s, t_0, t_1) \bigg|_{s = \lambda_i} \right) \bar{B}
\]
where
\[
U(s, t_0, t_1) \bigg|_{s = \lambda_i} =
\begin{bmatrix}
U(s, t_0, t_1) \bigg|_{s = \lambda_i} \\
\vdots \\
U(s, t_0, t_1) \bigg|_{s = \lambda_i}
\end{bmatrix}
\]
and
\[
U(s, t_0, t_1) \bigg|_{s = \lambda_i} =
\begin{bmatrix}
U & 0 & \cdots & 0 \\
\frac{\partial U}{\partial s} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
1 & \frac{\partial^{n-1} U}{\partial s^{n-1}} & \cdots & \frac{\partial U}{\partial s} & U
\end{bmatrix}
\]
Eqn. (9) is equivalent to Eqn. (1), the original differential equation of the system. Therefore, the analysis of a system characterized by a set of linear time-invariant differential equations undergoing state changes in finite time can be simplified to a problem involving a set of linear algebraic equations.

Open-loop feedback control:

The theory developed in the previous section can be utilized in the implementation of an open-loop feedback control [6]. Assuming that
\[
u(t) = q(1)^T K
\]
where
where $Q(s, b, \theta)$ is the Finite Laplace transform of $q(t)$.

Substituting Eqn. (14) into Eqn. (9)

$$EXF_t^a = E_t^x + (Q(s, t, b)T - X)K_f.$$  

Solving Eqn. (16) for $K$, we get

$$K = \hat{Q}(t_0, t)^{-1}C^{-1}[EXF_t^a(\hat{A}t) \bar{x}(t) - EXF_t^a(\hat{A}t) \bar{x}(t)].$$ (18)

Eqn. (18) can be written in a more general form as

$$K = \hat{Q}(t, t)^{-1}C^{-1}[EXF_t^a(\hat{A}t) \bar{x}(t) - EXF_t^a(\hat{A}t) \bar{x}(t)].$$ (19)

Substituting Eqn. (19) into Eqn. (12), the open-loop feedback control can be obtained as

$$u(t) = \hat{Q}(t, t)^{-1}C^{-1}[EXF_t^a(\hat{A}t) \bar{x}(t) - EXF_t^a(\hat{A}t) \bar{x}(t)].$$ (20)

Eqn. (19) clearly indicates that such a control exists if the matrix $\hat{Q}(t, t)$ is invertible for the defined interval of time. Furthermore, the open-loop feedback control is equivalent to a linear state feedback control with time-varying feedback coefficient.

Example:

Consider the system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 2 \end{bmatrix} x(t)$$

with the given boundary conditions

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

and

$$x(1) = \begin{bmatrix} -1.5 \\ 1 \end{bmatrix}$$

The transfer function of the system can be obtained as

$$\frac{Y(s)}{U(s)} = \frac{2s}{s^2 + 2s + 2}$$
with the boundary conditions
\[ y(0) = 0, \]
\[ y(0) = 0, \]
\[ y(1) = 2, \]
and
\[ y(1) = 0. \]
The Jordan canonical form of the system can be derived as
\[ \bar{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \]
\[ y = \begin{bmatrix} -2 & 4 \end{bmatrix} \bar{x}(t) \]
with the boundary conditions
\[ \bar{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \]
and
\[ \bar{x}(1) = \begin{bmatrix} -2 \\ -0.5 \end{bmatrix}, \]
where
\[ x(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bar{x}(t). \]

Suppose the selected control has the form of
\[ u(t) = K_1 + K_2 t = \begin{bmatrix} 1 \\ t \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}. \]
Then, Eqn. (16) can be found as
\[ \begin{bmatrix} e & 0 \\ 0 & e^2 \end{bmatrix} \begin{bmatrix} -1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} e^{-1} \\ e^2-1 \\ 2e^2+1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}. \]

Therefore,
\[ \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} -18.837 \\ 26.931 \end{bmatrix}, \]
which means
\[ u(t) = -18.837 + 26.931 t. \]

In addition, substituting
\[ x(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} x(t) \]
into Eqn. (20), the open-loop feedback can be obtained as
\[ u(t) = \begin{bmatrix} 1 \\ t \end{bmatrix} \begin{bmatrix} e - e^t \\ e^2 - e^{2t} \end{bmatrix} \begin{bmatrix} e(1 - t) \\ e^2 + e^{2t} - 2te^{2t} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ e^2 \end{bmatrix} \begin{bmatrix} -2 \\ -0.5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} x(t). \]

Conclusion:
In this paper, the entire set of solution to the two-point-boundary-value problems was obtained. It was shown that the open-loop feedback control is equivalent to a linear state feedback control with time-varying feedback coefficient.

References: