Abstract

A dynamic game approach to $H_\infty$ control for singularly perturbed systems is presented. Decomposition of system results in slow and fast subsystems which include the cost functions containing non-orthogonal terms. The fast subproblem can be considered as one of special cases of the slow subproblem. A method for solving the $H_\infty$ control problems with cost functionals containing non-orthogonal terms is developed.

Introduction

One of the major developments in control theory that has received much attention in the last few years is the $H_\infty$ control problem [7]. However, the $H_\infty$ control problem for singularly perturbed systems has not yet been widely considered [11]. In the past, Vian and Sawan applied the state-space solution of the $H_\infty$ control problem [4] to the equivalent frequency-domain realizations of the slow and fast subsystem models to study state feedback control. Shahruz used the same state-space solution to approximate the solutions of the Riccati equations and then the approximate stabilizing compensators are designed for singularly perturbed systems [9]. In this paper, the zero-sum dynamic game theory which is mathematically and more encompassing than the above frequency-domain approach is applied to obtain the suboptimal state feedback controller for singularly perturbed $H_\infty$ control problem [1,10]. The problem of existence of the cross terms between disturbance, the control, and the state in the performance index of the slow subproblem which had been ignored in the past can be easily handled using the dynamic game approach.

Preliminaries

Consider the linear time invariant system:

$$\dot{x} = Ax + Bu + Fw \quad ; \quad x(0) = 0 \tag{1}$$
$$y = Cx + Du + Ew \tag{2}$$

where

$\dot{x}(t)\in\mathbb{R}^n$ is the state vector
$u(t)\in\mathbb{R}^m$ is the controlled input
$w(t)\in\mathbb{R}^r$ is disturbance
$y(t)\in\mathbb{R}^p$ is the system output
$A,B,C,D,E,F$ are constant matrices of the appropriate dimensions.

Assume $(A,B,C)$ is stabilizable-detectable, $(A,F)$ is stabilizable. $A$ is Hurwitz.

Define the performance index:

$$J = \frac{1}{2}\int_0^\infty \left( \| y \|^2 + \| y \|^2 \gamma^2 \right) \ dt \tag{3}$$

$\gamma$ is a given attenuation level.

Define

$$R = D^TD \tag{4}$$
$$V = \gamma^2I - E^TE \tag{5}$$
$$S = I - DR^{-1}D^T \tag{6}$$
$$W = I + EV^TE^T \tag{7}$$

Theorem

The two-person zero-sum dynamic game described by (1),(2) and (3) admits a unique feedback saddle point if

$$\gamma^2I - E^TE > 0 \tag{8}$$

Then the unique saddle-point strategies are given by

$$u^* = -(D^TD)^{-1}(D^TWC + B^TFK + D^TE(V^TSE)^{-1}FK) \tag{9}$$
$$w^* = (\gamma^2I - E^TE)^{-1}(E^TSC + F^TKE^TD^T(B^TK)^{-1}) \tag{10}$$

$K \geq 0$ is a solution of the algebraic Riccati equation:

$$KA_a + A_a^TK + Q_a - KR_aK = 0 \tag{11}$$

where

$$A_a = A - B(D^TD)^{-1}D^TW \tag{12}$$
$$Q_a = C^T[I + E(\gamma^2I - E^TSE)^{-1}ES] \tag{13}$$
$$D(D^TD)^{-1}D^T W \tag{14}$$
\[ R_s = R_s^\top = B(D^\top WD)^\top B^\top + B(D^\top WD)^\top D^\top EV^\top F^\top + [B(D^\top WD)^\top D^\top EV^\top F^\top]^\top - F(\gamma H-E^\top S)E^\top F^\top \]  

(14)

\[ u^* \text{ of } (9) \text{ assures that} \]

\[ \| T_{yw}(j\omega) \|_\infty \leq \gamma \]  

(15)

where \( T_{yw}(j\omega) \) denotes the transfer function from \( w \) to \( y \) and the infinity norm of transfer function is defined by

\[ \| T(j\omega) \|_\infty = \sup \sigma_{\text{max}}[T(j\omega)] \]  

(16)

\( \sigma_{\text{max}} = \) maximum singular value.

**Proof:** (see Appendix)

**case 1:** If \( E=0 \) then (11) becomes

\[ KA + A^\top K + K(BB^\top - \gamma^2 F^2)K = 0 \]  

(17)

The unique equilibrium strategies are given by

\[ u^* = -B^\top Kx \]  

(18)

\[ w^* = \gamma^2 F^\top Kx \]  

(19)

**case 2:** If \( E=0 \) and \( D^\top[C \ D] = [0 \ I] \), the ARE (11) is

\[ KA + A^\top K + C^\top C - K(BB^\top - \gamma^2 F^2)K = 0 \]  

(20)

which is the algebraic Riccati equation of the standard H\(_\infty\) problem [4]. In this case, the unique equilibrium strategies are given by

\[ u^* = -B^\top Kx \]  

(21)

\[ w^* = \gamma^2 F^\top Kx \]  

(22)

**Problem statement**

Consider the singularly perturbed system

\[ \dot{x} = A_{11}x + A_{12}z + B_1u + F_1w \]  

(23a)

\[ \dot{z} = A_{21}x + A_{22}z + B_2u + F_2w \]  

(23b)

\[ y = C_1x + C_2z + D_1u + E_1w \]  

(24)

where \( \varepsilon \) is a small positive parameter; \( x(t) \in R^n \), \( z(t) \in R^m \) are the states; \( u(t) \in R^m \), \( w(t) \in R^q \), \( y(t) \in R^p \) are the control, disturbance, and the system output, respectively. \( A_{ij}, B_i, C_i, F_i, D_i \) for \( i=1,2 \) and \( j=1,2 \) are constant matrices of the appropriate dimensions. \( A_{ij} \) is nonsingular.

Find a state feedback controller

\[ u = G_1x + G_2z \]  

(25)

such that the infinity norm of the transfer function from \( w \) to \( y \), ie. \( T_{yw} \), is

\[ \| T_{yw}(j\omega) \|_\infty \leq \gamma \]  

(26)

where \( \gamma \) is a given attenuation level.

In order to make the mapping \( w \to y \) linear, the initial states are assumed to be zeros.

Using the link between the H\(_\infty\) optimal control and the linear quadratic differential game theory [1], we denote:

\[ J = \frac{1}{2} \int \left( \gamma^2 \| y \|_2^2 - \gamma^2 \| w \|_2^2 \right) dt \]  

(27)

We find \( u^* \) and \( w^* \) that brings \( J \) to a saddle point equilibrium. The optimal control, \( u^* \), forces the system (23),(24) to satisfy (26).

**Slow subsystem**

Let \( \varepsilon \to 0 \) in (23), we have:

\[ \dot{x} = A_{11}x + A_{12}z + B_1u + F_1w \; ; \; \dot{x}(0) = x(0) \]  

(28)

\[ 0 = A_{21}x + A_{22}z + B_2u + F_2w \]  

(29)

\[ y = C_1x + C_2z + D_1u + E_1w \]  

(30)

We substitute (29) into (28) and (30), we define the slow subsystem:

\[ \dot{x}_s = A_{11}x_s + B_1u_s + F_1w_s \; ; \; x_s(0) = x(0) \]  

(31)

\[ y_s = C_1x_s + D_1u_s + E_1w_s \]  

(32)

where

\[ A_{11} = A_{11} - \varepsilon A_{12}(A_{22})^{-1}A_{21} \]  

(33)

\[ B_1 = B_1 - \varepsilon A_{12}(A_{22})^{-1}B_2 \]  

(34)

\[ F_1 = F_1 - \varepsilon A_{12}(A_{22})^{-1}F_2 \]  

(35)

\[ C_1 = C_1 - \varepsilon C_2(A_{22})^{-1}A_{21} \]  

(36)

\[ D_1 = D_1 - \varepsilon C_2(A_{22})^{-1}B_2 \]  

(37)

\[ E_1 = -\varepsilon C_2(A_{22})^{-1}F_2 \]  

(38)

The performance index [2]:

\[ J_s = \frac{1}{2} \int_0^\infty (\gamma^2 \| y_s \|_2^2 - \gamma^2 \| w_s \|_2^2) dt \]  

(39)

We assume that \( (A_{11}, B_1, C_1) \) is stabilizable-detectable and \( (A_{11}, F_1) \) is stabilizable [8].

The state feedback control that minimizes the infinity norm transfer function of the slow subsystem is

\[ u_s^* = (D_1^\top W D_1)^{1/2} [D_1^\top W C_1 + B_1^\top K_s] \]

\[ + D_1^\top E V^\top F^\top K_s x_s \]  

(40)

\[ K_s \geq 0 \] is a solution of the algebraic Riccati equation:

\[ K_s A_{11} + A_{11}^\top K_s + Q_s - K_s R K_s = 0 \]  

(41)

where

\[ A_{11} = A_{11} - B_1(D_1^\top WD_1)^{1/2} D_1^\top W C_1 \]

\[ + F_1(\gamma^2 H-E^\top S)E^\top F_1^\top \]  

(42)
Let $T_{ij}(j\omega)$ be transfer function from $w_i$ to $y_{ij}$. 

\begin{align}
Q_{ij} &= Q_{ij}^T = C_i H + E_i (\gamma I - E_i S_i E_i)^{-1} E_i S_i , \\
R_{ij} &= R_{ij}^T = B_i (D_i^T W_i D_i + B_i^T E_i V_i F_i) + [B_i (D_i^T W_i D_i + B_i^T E_i V_i F_i)]^T \\
& - F_i (\gamma I - E_i S_i E_i)^{-1} F_i^T 
\end{align}

(43)

(44)

and 

\begin{align}
R_i &= D_i^T D_i , \\
V_i &= \gamma I - E_i E_i , \\
S_i &= I - D_i R_i^T D_i^T , \\
W_i &= 1 + E_i V_i E_i^T 
\end{align}

(45)

(46)

(47)

(48)

Let $T_{ij}(j\omega)$ be transfer function from $w_i$ to $y_{ij}$. $u_i^*$ of (40) 

leads to 

\[ \|T_{ij}(j\omega)\|_\infty \leq \gamma \] 

(49)

Fast subproblem

The slow variables are assumed to be constant during the fast transient. Define 

\begin{align}
x &= \bar{x} + x_s , \\
z &= \bar{z} + z_s , \\
u &= \bar{u} + u_s , \\
w &= \bar{w} + w_s , \\
y &= \bar{y} + y_s 
\end{align}

(50)

Using (50) into (23)(24), we define the fast subsystem: 

\begin{align}
\dot{z}_s &= A_s z_t + B_s u_s + F_s w_s \\
y_t &= C_s z_t + D_s u_s 
\end{align}

(51)

(52)

The performance index [2] : 

\[ J_s = \frac{1}{2} \int_0^\infty (\|y_t\|_2^2 + \dot{z}_s^2 + w_s^2) \, dt \] 

(53)

We assume that $(A_{22}, B_{22}, C_2)$ is stabilizable-detectable and $(A_{22}, F_2)$ is stabilizable [8].

The state feedback control that minimizes the infinity norm of the transfer function of the fast subsystem is 

\[ u_s^* = -R_s^{-1}(D_{22} C_2 + B_{22} K_{22}) z_t \] 

(54)

$K_{22} \geq 0$ is the solution of the algebraic Riccati equation: 

\[ K_s A_s + A_s^T K_s + Q_s - K_s R_s K_s = 0 \] 

(55)

where 

\begin{align}
R &= D_D^T D_D \\
A_s &= A_{22} B_s R_s^{-1} D_s C_s \\
Q_s &= Q_s^T = C_s (I - D_s R_s^{-1} D_s^T) C_s \\
R_s &= R_s^T = B_s R_s^{-1} B_s^T \gamma^{-2} F_s F_s^T 
\end{align}

(56)

(57)

(58)

(59)

Let $T_{ij}(j\omega)$ be transfer function from $w_i$ to $y_{ij}$. $u_i^*$ of (54) 

gives 

\[ \|T_{ij}(j\omega)\|_\infty \leq \gamma \] 

(60)

Composite control

Let 

\begin{align}
G_s &= (D_i^T W_i D_i + B_i^T E_i V_i F_i)^{-1} (D_i^T W_i C_i + B_i^T K_s) \\
G_r &= (\gamma I - E_i E_i)^{-1} (E_i S_i C_i + F_i K_s) \\
G_u &= (\gamma I - E_i E_i)^{-1} (E_i S_i E_i + F_i^T F_i) 
\end{align}

(61)

(62)

(63)

The composite state feedback control that forces the transfer function, $T_{ij}$, of (23)(24) to satisfy (26) is 

\begin{align}
u_s^* &= u_s^* + u_i^* \\
 &= G_s x_s + G_r z_t \\
 &= G_s x + G_r z \end{align}

(64)

where 

\[ G_s = [I + G_s (A_{22}) B_s] G_s + G_s (A_{22}) + F_s G_u \] 

(65)

Let $T_{ij}(j\omega)$ be transfer function from $w$ to $y$, we have 

\[ \|T_{ij}(j\omega)\|_\infty \leq \gamma \] 

(66)

Example

Consider the following system 

\begin{align}
\dot{x} &= -2x + 2u + w \\
\dot{z} &= -2x + z + u + w \\
y &= x + z + u \\
y &= 2x + z + u
\end{align}

(67)

The performance index 

\[ J_s = \frac{1}{2} \int_0^\infty (\|y\|_2^2 + \|z\|_2^2 + \|w\|_2^2) \, dt \] 

(68)

where $\gamma = 0.01$, $\gamma = 50$, and $y = [y_1, y_2]^T$

Slow-Fast problem 

The composite state feedback controller is 

\[ u_s^* = G_s x + G_r z \] 

(69)

where 

\[ [G_s, G_r] = [-2.17155868, -1.82847568] \] 

(70)

Then 

\[ \|T_{ij}(j\omega)\|_\infty = 0.821284726 \] 

(71)

Full-order problem 

The optimal controller is 

\[ u^* = F_s x + F_r z \] 

(72)

where 

\[ [F_s, F_r] = [-2.171355608, -1.828678902] \] 

(73)

and 

\[ \|T(j\omega)\|_\infty = 0.821261776 \] 

(74)
Conclusion

The suboptimal controller for singularly perturbed $H_\infty$ control problem is derived using the zero-sum dynamic game approach. This time domain approach seems to be very simple and gives acceptable results. The above example shows that the order of error is $O(c^2)$. The finite horizon and the time-varying versions of the original problem can be handled using this approach so that the transient behavior of system can be studied [1].

Appendix

For the system (1),(2), and (3), the Hamiltonian function is defined [6] :

$$H = \frac{1}{2} \| y \| \dot{y}^2 + \frac{1}{2} \| w \| \dot{w}^2 + \pi^T \dot{x}$$  (A1)

$$= \frac{1}{2} \| x \| \dot{x}^2 + \frac{1}{2} \| u \| \dot{u}^2 + \pi^T \dot{x} + \pi^T \dot{w}$$  (A2)

Therefore, the game described (1),(2),(3) admits a unique equilibrium point if

$$\gamma I - E^T E > 0$$  (A3)

and of (A2) and (A3) can be determined from

$$A = Ax + Bu + Fw$$  (A4)

$$\dot{p} = -C^T (Cx + Du + Ew) - A^T \pi$$  (A5)

Let $A = D^T W D$ and $D = \gamma I - E^T E$  (A6)

$$\dot{r} = -C^T (Ax + Bu + Fw) - A^T \pi$$  (A7)

$$\dot{p} = -C^T (D^T W D - E^T E) x - A^T \pi$$  (A8)

Substitute u and w of (A2),(A3) into (A5),(A6), we have

$$\dot{x} = [A - B A \Delta^T D^T W C + FA^T E^T E] x + [FA^T (F^T E^T D^T B^T - B A \Delta^T D^T W C)] \pi$$  (A9)

$$\dot{p} = C^T [D^T W I - E A^T E] x - A^T \pi \dot{x}$$  (A10)

Since $A = \gamma I - E^T E$

$$\Rightarrow \quad \Lambda = V + E^T D^T E$$

$$\Rightarrow \quad \Lambda^T = \Lambda + V^T D^T E$$

$$\Rightarrow \quad D^T E^T E = D^T E A^T E + \Delta E^T E D^T E E^T$$

$$\Rightarrow \quad D^T [I + \Delta E^T E] = D^T W D^T E A^T E + R^T$$

$$\Rightarrow \quad D^T W = \Delta R^T D^T E A^T E + D^T$$

$$\Rightarrow \quad \Delta D^T W = R^T D^T E A^T E + \Delta D^T$$  (A11)

Similarly, we have

$$(\Lambda^T E^T S)^T = E A^T - D \Delta^T D^T E^T$$  (A12)

$$R^T D^T E A^T = \Delta D^T D^T E^T$$  (A13)

**Lemma** (A14)

(DA^T D^T W - EA^T E^T S is symmetry)

$$R^T D^T E A^T = \Delta D^T D^T E^T$$

since (4-7).

The minimum principle shows that the necessary condition for the control and disturbance to optimize the cost (3) : 

$$D^T [Cx + Du + Ew] + B^T \pi = 0$$  (A2)

$$E^T [Cx + Du] + (E^T E - \gamma I) w + F^T \pi = 0$$  (A3)

Therefore, the game described (1),(2),(3) admits a unique equilibrium point if

$$\gamma I - E^T E > 0$$  (A4)

The minimum principle shows that the necessary condition for the control and disturbance to optimize the cost (3) : 

$$D^T [Cx + Du + Ew] + B^T \pi = 0$$  (A2)

$$E^T [Cx + Du] + (E^T E - \gamma I) w + F^T \pi = 0$$  (A3)

Therefore, the game described (1),(2),(3) admits a unique equilibrium point if

$$\gamma I - E^T E > 0$$  (A4)

and p of (A2) and (A3) can be determined from

$$\dot{x} = Ax + Bu + Fw$$  (A5)

$$\dot{p} = -C^T (Cx + Du + Ew) - A^T \pi$$  (A6)

Let $A = D^T W D$ and $D = \gamma I - E^T E$  (A7)

$$\dot{r} = -C^T (Ax + Bu + Fw) - A^T \pi$$  (A8)

$$\dot{p} = -C^T (D^T W D - E^T E) x - A^T \pi$$  (A9)

Equations (A9),(A10) become :

$$\dot{x} = A_1 x - R_1 \pi$$  (A11)

$$\dot{p} = -Q_1 x - A_1 \pi$$  (A12)

Let $p = K x$ then (A20) and (A21) can be formed as:

$$K A_1 + A_1^T K + Q_1 - K R_1 K = 0$$  (A22)

Since the control $u'$ minimizes J and disturbance w' maximizes J, it is clearly to say that:

$$J(u',w') \leq J(u,w)$$  (A23)
$J(u^*,w^*)$ is the value of the saddle point equilibrium.

From (3) and (A23), we have:

$$
\|y(u^*,w)\|_2^2 - \gamma^2 \|w\|_2^2 \leq \|y(u^*,w^*)\|_2^2 - \gamma^2 \|w^*\|_2^2 \quad (A24)
$$

Let $x^*(t)$ be the solution of the closed loop system, where

$$
\dot{x}^* = [A + B\Delta^r(D^TW + B^T K + D^T E V^{-1} F^T K) + F A^r (E^T S C + F^T K - E^T D R^B B^T K)]x^* + (C - \Delta^r(D^TW + B^T K + D^T E V^{-1} F^T K) + E A^r (E^T S C + F^T K - E^T D R^B B^T K))x^*
$$

The solution of the closed loop system (A25) is $x^*(t) = \phi(t)x^*(0)$, $\phi(t)$ is the transition matrix. Therefore, $x^*(t) = 0$ then $y(u^*,w^*) = 0$ and $w^* = 0$.

The inequality (A24) becomes:

$$
\|y(u^*,w)\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0 \quad (A25)
$$

The inequality (A24) is equivalent to:

$$
\max_{u^*} \frac{\|y(u^*,w)\|_2^2}{\|w\|_2^2} \leq \gamma \quad (A29)
$$

Therefore [3,5],

$$
\|T_m\|_{\infty} \leq \gamma \quad (A30)
$$

Thus, $u^*$ provides the $H_\infty$ optimal control policy.

References


