THE WAVE PACKET TRANSFORM (WPT) AS APPLIED TO SIGNAL PROCESSING

Theodore E. Posch
Hughes Aircraft Company, 1901 W. Malvern Ave., Fullerton, CA 92634 USA

ABSTRACT
The Wave Packet Transform (WPT) uses the Weyl operator and wave packet functions, i.e., functions that are similar to wavelets, in a linear form to compute coefficients in a two dimensional space of time and frequency. We discuss the importance of the Weyl operator in the WPT and its use with different wave packets. We show that the energetic form of the WPT the wavepacketgram, i.e., the modulus square of the WPT is a member of Cohen's class of time-frequency distributions which we call the wave packet Cohen class distribution. We show that the wavepacketgram is a positive time-frequency distribution.

1. INTRODUCTION
Two well established linear transforms are currently used in signal processing. One is the STFT which uses the Weyl operator and the Fourier kernel familiar to all of us who have used the short time Fourier transform. The other is the wavelet transform which uses dilation and translations of a set of functions that are complete and indeed in some cases over complete. Both of these transforms can be represented using an inner product formulation. The wave packet transform (WPT) is a linear transform which uses the Weyl operator and a set of functions called wave packets. Wave packets do resemble wavelets since they are dilations of some well localized function, and wavelets can be used as window functions in the WPT. The WPT is the Fourier transform of a signal windowed with a wavelet that is dilated by a and displaced by b. The basis functions are modulated versions of the wavelet window function.

A formal mathematical discussion on Wave Packets and Wave Fronts can be found in Chapt. 3 in a book by Folland [1] on “Harmonic Analysis in Phase Space”. The mathematical formalism in this book is presented in phase space notation, however, position momentum space notation is equally applicable to time frequency space notation. The idea of using concepts that have been developed in phase space and translating them to concepts in time frequency space is not really new and it has been done by numerous authors as can be seen in the recent review article by Cohen [2].

One major objective in signal processing is to localize a signal in some two dimensional space using a basis function that best represents the signal. For example, if our signals are stationary over some short interval of time, then the short time Fourier transform using a window and the spectrogram does the best job in localizing the signal in the time frequency plane. This does not mean that the spectrogram is the correct joint time-frequency representation, however it has been easy to interpret. If the signal or signals are of extremely short duration or time varying then the Cohen class time frequency distribution using a kernel function that preserves time support over the analysis interval would seem to be the most appropriate for time-frequency localization. Recently, Daubechies [3] and many others [4] have proposed using wavelets for the analysis of short duration signals. Here we use the zoom-in capability of the wavelet as a window function that changes its time support as a function of the scale in the WPT.

2. THE WAVE PACKET TRANSFORM USING THE WEYL OPERATOR
The term “wave packet” is a function \( \phi \) on \( \mathbb{R}^n \) that is well localized in phase space, that is \( \phi \) and its Fourier transform are both concentrated in reasonably small sets. The most typical example of wave packets are the Gaussians \( \phi(t) = e^{2\pi i f t} e^{-\alpha(t-t_0)^2} \).

A family of wave packets can be obtained by subjecting the selected wave packet to dilations such as \( \phi^a(t) = a^{n/4} \phi(a^{1/2}t) \), where \( a \geq 0 \), and then to translations in time frequency space by using the Weyl oper-
ator, \( \psi^{\alpha}(t) = \alpha^{n/4}e^{-j\alpha t}e^{{j\alpha}^2t}(t - \tau) \).

Wavelets can be used as wavepackets since they have a dilation factor and all we have to remember is that the time translation using the Weyl operator has already been taken into account in the wavelet. Note that the sign in the operator is negative. Normally, it is customary to keep the signs the same, since that is the way they are used when one is using the characteristic function. We shall indicate in the paper where we make a change in the signs. For example, if we define the operator acting on the wave packet with a positive sign we would obtain \( \psi^{\alpha}(t) = \alpha^{n/4}e^{-j\alpha t}e^{{j\alpha}^2t}e^{(\alpha/2)(t - \tau)} \).

The formal mathematical definition of the WPT with the Weyl operator and a negative sign in the operator and a wave packet for the window function is

\[
P_{\psi}^{\alpha}(\theta, \tau, a) = \int f(t) e^{j2\pi(\theta T - t \tau)} \psi^{\alpha}(t) dt
\]

We shall use the notation used in (2) for operators in order to be consistent with the time frequency literature. In Eq. (2) the terms in the exponent are operators in time-frequency space, \( \psi^{\alpha} \) is the wave packet, and the overbar indicates complex conjugate. All integrals are from \(-\infty \) and \( \infty \) unless otherwise noted. When \( f = \psi \) we call the transform the auto-WPT. The operators are time and frequency operators and are represented by the symbols

\[
T \rightarrow t \quad F \rightarrow -j2\pi \frac{d}{dt} \quad (time \ domain)
\]

\[
T \rightarrow j2\pi \frac{d}{dq} \quad F \rightarrow f \quad (frequency \ domain)
\]

These operators do not commute and satisfy the fundamental commutation relation

\[
[T, F] = TF - FT = j
\]

The operators that we shall use are linear and Hermitian. The Hermitian property states that for any two functions and a Hermitian operator we have that

\[
\int \overline{f(t)}O(T, F; \theta, \tau) \psi^{\alpha}(t) dt = \int \psi^{\alpha}(t)O^*(T, F; \theta, \tau) \overline{f(t)} dt
\]

The generalized form of the Wave Packet Transform (WPT) can now be defined as

\[
\int \overline{f(t)}O(T, F; \theta, \tau) \psi^{\alpha}(t) dt
\]

where the signal is \( f(t) \), the operator is \( O(T, F; \theta, \tau) \) and the wave packet is \( \psi^{\alpha}(t) \).

It is worthwhile at this point to compare the short time Fourier transform (STFT), the wavelet transform (WT), and the wave packet transform (WPT). The continuous STFT is defined as

\[
S(\tau, f) = \int_{-\infty}^{\infty} e^{j2\pi ft} g(t - \tau) f(t) dt
\]

and the continuous WT is defined as

\[
W(b, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} \phi \left( \frac{t - b}{a} \right) f(t) dt
\]

and the continuous WPT using a wavelet window is defined as

\[
WP(b, f, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{j2\pi ft} \phi \left( \frac{t - b}{a} \right) f(t) dt
\]

The wavelet transform uses a scale length variable \( a \). The inverse relation would be scale number variable \( s \) where \( a = \frac{1}{s} \). The standard convention used for the remaining variables are frequency and time both of these variables have some physical relation that can be measured.

It is important to see that the WPT is a function of time, frequency, and scale length. We have used the wave packets to analyze signals in the time frequency plane similar to the way the orthogonal wavelets are used in the wavelet transform to analyze the signal in terms of time and frequency octaves, however we have introduced a modulation function to localize the frequency content of the signal within an octave. One form of the wavelet transform uses a bank of bandpass filters that are scaled versions of the original filter, hence producing a bank of constant Q filters. The WPT using wavelets for a particular choice of wave packets uses the same bank of filters as used in the wavelet transform, i.e., filters that have a constant relative bandwidth. The next step in computing the WPT is to take the FFT of the constant Q filter bank outputs to obtain the localization of the signal within the octave bands.

3. WAVELETS AS WINDOWS IN THE WPT
Windows are the functions we use to observe nature. Nature does not tell us to use short duration windows or long duration windows to observe signals. If we happen to get the wrong answer because we used the wrong window it is our fault not nature's. Isn't it a curious fact that the way we perceive nature as human beings is to use windows associated with our receptors such as the classical lateral inhibition function used in our visual, auditory, and skin sensors. Therefore we should not be too surprised that the suggestion of using wavelets as windows to look at nature has a close if not an identical resemblance to the transfer function for lateral inhibition function.

In 1945, D. Gabor [5] reviewed the analysis of hearing as it was known that day by using his concept of "logons" time-frequency cells in the time-frequency plane that were used to describe joint time-frequency signals. The signals that Gabor proposed is, "the signal which occupies the minimum area $\Delta t \Delta f = 1/2$ is the modulation product of a harmonic oscillation of any frequency with a pulse of the form of a probability function. In complex form $p(t) = e^{-\alpha^2(t-\tau)^2} e^{2\pi f(t)}$ ...". The name of the inner product of this function with the signal is now called the Gabor transform.

4. WAVE PACKETS IN GENERAL

A typical family of wave packets can be made by subjecting $\phi$ to dilations such as $\phi^\alpha(t) = e^{-\alpha^2(t-\tau)^2} e^{2\pi f(t)}$. Using Eq. (1) with the variable in the wave packet $\alpha = 1$ we obtain

$$P^\phi_\alpha(\theta, \tau, 1) = \int \mathcal{F} \left( e^{-\alpha^2(t-\tau)^2} e^{2\pi f(t)} \right) dt$$

We can write the operators in the exponent as

$$e^{2\pi f(t-\tau) \alpha} = e^{-\alpha^2(t-\tau)^2} e^{2\pi f(t-\tau)}$$

Now, using the Baker-Hausdorff theorem we can write Eq. (1) for the case where $\alpha = 1$ as

$$P^\phi_\alpha(\theta, \tau, 1) = \int \mathcal{F} \left( e^{-\alpha^2(t-\tau)^2} e^{2\pi f(t-\tau)} \right) dt$$

One can immediately see that the WPT using the Weyl operator is simply a phase term multiplied by the short term Fourier transform (STFT) for the case when $\alpha = 1$. In Eq. (14) we make a change in the variables and let $t = y + \tau/2$, we obtain the narrowband cross ambiguity function

$$P^\phi_\alpha(\theta, \tau, 1) = \int \mathcal{F} \left( e^{-\alpha^2(t-\tau)^2} e^{2\pi f(t-\tau)} \right) dt$$

An interesting alternative way to write Eq. (1) is as a convolution which we shall discuss because of the utility of convolution in signal processing. This is a modification to the procedure that was done by Folland [1]

$$P^\phi_\alpha(\theta, \tau, \alpha) = \int \mathcal{F} \left( e^{-\alpha^2(t-\tau)^2} e^{2\pi f(t-\tau)} \right) dt$$

In other words we can write the WPT as

$$P^\phi_\alpha(\theta, \tau, \alpha) = e^{j\alpha \tau} \left( \mathcal{F} \left( f \psi_{\tau, \alpha}(t) \right) \right)$$

In Eq. (19) we see that the WPT can be obtained by taking the Fourier transform of the conjugated signal multiplying it by a phase term or even neglecting the phase term and convolving with Eq. (17) the Fourier transform of the wavelet function.

$$\mathcal{F} \rightarrow \mathcal{F} \left( \mathcal{F} \left( f \psi_{\tau, \alpha}(t) \right) \right)$$

5. THE MODULUS SQUARE OF THE WPT IS A MEMBER OF THE COHEN CLASS

When one is using the Cohen formulation to describe time frequency distributions it should be clearly understood that in using the Cohen class time frequency distribution that the selection of the kernel results in different time frequency distributions. We can write the Cohen class time-frequency distribution for signals $f(u)$ as

$$P(t, f) = \int \phi(t, f) f(u + \tau/2) \phi^*(u - \tau/2)$$

$$\times e^{-\alpha^2(t-\tau)^2} e^{2\pi f(t-\tau)}$$

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and \( \phi(\theta, \tau) \) is the kernel function.

Perhaps the simplest way to define the Wavepacketgram is to say that it is the modulus square of the WPT. This is somewhat analogous to the way the spectrogram is defined as the modulus square of the STFT. We now show that the Wavepacketgram is a member of the Cohen class time-frequency distributions where the kernel is the ambiguity function of the wavelet window used in the WPT.

We can use the same procedure that was used for the STFT and the spectrogram for the WPT and the Wavepacketgram, respectively. Using time-frequency-scale notation for the WPT we express it as

\[
W(b, f, a) = \frac{1}{a} \int x(t) \phi \left( \frac{t-b}{a} \right) e^{-j2\pi ft} dt 
\]

and write the Wavepacketgram the modulus square of the WPT as

\[
|W(b, f, a)|^2 = \frac{1}{a} \int x(t) \phi \left( \frac{t-b}{a} \right) e^{-j2\pi ft} dt \times \frac{1}{a} \int x(t') \phi \left( \frac{t'-b}{a} \right) e^{j2\pi ft'} dt'
\]

Now, let \( v = (t - t') \) and \( u = \frac{1}{2} (t + t') \) so that \( t = u + \frac{v}{2} \) and \( t' = u - \frac{v}{2} \).

So the wavepacketgram can be rewritten as

\[
|W(b, f, a)|^2 = \frac{1}{a} \int e^{j2\pi ft} x \left( u + \frac{v}{2} \right) x \left( u - \frac{v}{2} \right) \phi \left( \frac{u - \frac{v}{2}}{a} \right) \phi \left( \frac{u + \frac{v}{2}}{a} \right) dudv
\]

Define

\[
\phi \left( a\theta, \frac{v}{a} \right) = \int \phi \left( y - \frac{v}{2a} \right) \phi \left( y + \frac{v}{2a} \right) e^{-j2\pi a\theta y} dy
\]

Then

\[
\phi \left( \frac{u - \frac{v}{2}}{a} \right) \phi \left( \frac{u + \frac{v}{2}}{a} \right) = \int \phi \left( a\theta, \frac{v}{a} \right) d\theta \times e^{j2\pi \theta (u - \frac{v}{2})} d\theta
\]

Now substitute Eq. (26) into Eq. (24) to obtain

\[
|W(b, f, a)|^2 = \int \int x \left( u + \frac{v}{2} \right) x \left( u - \frac{v}{2} \right) \phi \left( a\theta, \frac{v}{a} \right) \times e^{j2\pi \theta (u - \frac{v}{2} - ft)} d\theta dudv
\]

We have shown that the Wavepacketgram is a member of the Cohen class time-frequency distributions where the kernel is a scale dependent ambiguity function. The Wavepacketgram is positive because it is the modulus square of the WPT. In order to visualize the Wavepacketgram because it has 3 variables of the left hand side of the equation we can associate the scale variable with frequency in an inverse relationship.

**CONCLUSION**

We have introduced the wave packet transform. The most significant result of this paper is the introduction of the wavelet function as a window function such that the signal can be analyzed with different window widths as a function of frequency in the WPT. We have generalized the WPT using different windows and operators.

We have obtained the wave packet form of the Cohen class of time-frequency distribution by using the modulus square of the WPT to write the Wavepacketgram. Here we show that the Wavepacketgram is the product of two ambiguity function one of the signal and the other of the wavelet window. We have shown that the Wavepacketgram is a positive time-frequency-scale distribution.

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**REFERENCES**