ON BLIND CHANNEL ESTIMATION WITH PERIODIC MISSES AND EQUALIZATION OF PERIODICALLY VARYING CHANNELS

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Abstract — Time series with systematic misses and signals passing through rapidly fading channels can be modeled as cyclostationary output sequences of linear periodically time-varying (LPTV) systems. Blind identification of finite parameter LPTV systems can be accomplished using cyclic second- and higher-order statistics of the cyclostationary output, even if the latter is corrupted by stationary noise of unknown spectral characteristics. Novel closed form solutions for the modulating sequence which models periodic misses, and the TV moving average parameters of periodic channels are developed. Using the estimated LPTV channel, the Viterbi algorithm, or, a computationally efficient LPTV mean-squared error equalizer can be adopted to recover the information bearing input.

1 Introduction

Consider the discrete-time model of Figure 1

\[ u(t) = \sum_{\tau=0}^{\infty} h(t; \tau) w(t - \tau) + v(t), \]  

and assume that: (as1) the information signal \( w(t) \) is zero-mean, i.i.d., non-Gaussian with finite moments, (as2) the noise \( v(t) \) is stationary with unknown spectrum and independent of \( w(t) \), and (as3) the system \( h(t; \tau) \) is linear and (almost) periodically time-varying (LPTV). If separable, then \( h(t; \tau) = \mu(t) h_0(\tau) \), and according to (as3), \( \mu(t) \) is an almost periodic sequence; i.e., \( \mu(t) \) accepts a Fourier Series expansion (in the m.s.s.) with coefficients \( M(\alpha) = \lim_{T \to \infty} \frac{1}{T} \sum_{\tau=0}^{T-1} \mu(t)e^{-j\alpha t} \). In the non-separable case and if \( h(t; \tau) \) is strictly periodic with period, say \( N \), then \( h(t + N; \tau) = h(t; \tau), \forall t, \tau \in Z \), and its Fourier coefficient is given by \( \hat{H}(n; \tau) = N^{-1} \sum_{n=-N/2}^{N/2-1} h(t; \tau)e^{-jn\tau} \). Under (as1-3) process \( u(t) \) is \( k\)-th order (almost) cyclostationary [3]-[5], which means that its moments and cumulants up to order \( k \) are (almost) periodic functions of \( t \) (see also [6]). Given finite samples \( u(t), t = 0, \ldots, T-1 \), we wish to identify the LPTV system (blind channel estimation) and recover the input (equalization).

The LPTV setup in (1) is appropriate for rapidly fading mobile radio channels which model intersymbol interference due to multipath or bandwidth constraints [12]. The separable case is of special interest for representing amplitude fluctuations in sonar data [13], and modeling periodic misses [3], [8], [10]. The structured (here periodic) time evolution allows estimation of second- and higher-order output statistics (HOS), even when the latter vary rapidly with time, which precludes usage of common adaptive algorithms such as LMS or RLS.

In Section 2 we briefly review estimation of cyclic statistics [4], and use them to detect the cycles present in \( u(t) \). Section 2.1 also deals with the estimation of \( \mu(t) \) and \( h_0(\tau) \) in the separable case. In Section 2.2 we expand \( h(t; \tau) \) over the Fourier basis (see also [11]) and exploit cyclic HOS for its estimation. A mean-squared error (MSE) equalizer for discrete-time LPTV systems is described in Section 3 (see also [5] for the continuous case), and simulations are presented in Section 4.

2 Blind LPTV channel estimation

Our tools for identifying \( h(t; \tau) \) in (1) will be time-varying (TV) cumulants of the cyclostationary \( u(t) \). Let \( c_{ku}(t; \tau) = \text{cum}[u(t), u(t + \tau), \ldots, u(t + \tau_{k-1})] \) denote the \( k\)-th order TV cumulant and \( M_{ku}(t; \tau) = E\{u(t + \tau_1) \cdots u(t + \tau_{k-1})\} \) the corresponding TV moment which due to (as3) can be expanded as:

\[ m_{ku}(t; \tau) = \sum_{\alpha} M_{ku}(\alpha; \tau)e^{j\alpha t}. \]

The so-called cyclic moment \( M_{ku} \) can be consistently estimated using [4]

\[ \hat{M}_{ku}(\alpha; \tau) = \frac{1}{T} \sum_{t=0}^{T-1} u(t)u(t + \tau_1) \cdots u(t + \tau_{k-1})e^{-j\alpha t}. \]

Because cumulants can be expressed in terms of moments [1], \( \hat{M}_{ku}(\alpha; \tau) \) in (2) can be used to form sample TV cumulants \( \hat{c}_{ku}(t; \tau) \); e.g., for \( k = 3 \) and...
since \( u(t) \) is zero mean,

\[
\hat{c}_{2u}(t; \tau_1, \tau_2) = \sum_{\alpha} \hat{C}_{2u}(\alpha; \tau_1, \tau_2) e^{j\alpha t},
\]

where \( \hat{C}_{2u}(\alpha; \tau_1, \tau_2) \) is obtained as in (2) with \( k = 3 \) and \( \alpha \) denotes the cycles for which \( \hat{C}_{2u}(\alpha; \tau_1, \tau_2) \neq 0 \).

Stationary process have time invariant cumulants and hence vanishing cyclic cumulants if all disturbances are suppressed. In addition, for a stationary process, the sample estimates.

\[
\hat{c}_{1u}(a; z) \quad \text{and} \quad \hat{c}_{2u}(a; z)
\]

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In summary, cyclostationarity provides an ensemble (of periods) for estimating TV statistics and a means of extracting cyclostationary signals corrupted by additive stationary noise.

2.1 Separable case – missing observations

Here (1) reduces to the multiplicative plus additive noise model

\[
u(t) = \mu(t)x(t) + v(t),
\]

where \( x(t) = \sum_{\tau=0}^{N-1} h_0(\tau) u(t - \tau) \) is stationary, and when \( \mu(t) \) is given by

\[
\mu(t) = \begin{cases} 
1, & x(t) \text{ measured}, \ t \in [0, N_1) \ (\text{mod} \ N) \\
0, & x(t) \text{ missed}, \ t \in [N_1, N_1 + N_0),
\end{cases}
\]

it implies that \( x(t) \) is periodically observed (in noise) for \( N_1 \) consecutive samples, and missed for the next \( N_0 \) samples. Subsequently, we develop a 3-step algorithm for (s1) determining the period \( N = N_1 + N_0 \), (s2) recovering \( \mu(t) \) along with \( N_1 \) and \( N_0 \), and (s3) estimating the channel \( h_0(\tau) \).

(s1) Estimating the period \( N \) – cycles \( \alpha \)

By definition it follows that \( C_{2u}(\alpha; 0) = \sigma_a^2 \lim_{N \to \infty} \frac{1}{N \sum_{T=0}^{N-1} \mu^2(t) e^{-j2\alpha t}} \), which implies that the fundamental frequency of \( C_{2u}(\alpha; 0) \) coincides with that of \( \mu^2(t) \). If \( \mu(t) = e^{j\tau_0 t} \), \( C_{2u}(\alpha; 0) \) will peak at \( \alpha = 2\tau_0 \); but if \( \mu(t) \) is generally periodic with period \( N \), \( C_{2u}(\alpha; 0) \) will peak at \( \alpha = 2\pi n/N \). These observations suggest the following empirical test for detecting the period in \( \mu(t) \):

\[
\hat{\alpha} = \arg \min_{0 < \alpha < \pi} \left\{ \alpha : |\hat{C}_{2u}(\alpha; 0)| \text{ is local max.} \right\},
\]

where \( \alpha > 0 \) is set to avoid the d.c. term which is affected by \( v(t) \). A similar frequency–domain peak-finding approach was suggested in [9], and a rigorous Neyman-Pearson statistical test was reported in [5].

When the input \( u(t) \) is complex and symmetrically distributed, \( c_{2u}(\tau) = c_{2u}(\tau, \tau) = 0 \). This may happen when \( u(t) \) corresponds to a QAM signal \( a(t) + j\beta(t) \) where \( a(t), \beta(t) \) are independent Gaussian variables. One approach is to use conjugated \( c_{1u}(\tau) = E\{u^*(t)u(t + \tau)\} \), but again with some simplifications, \( \mu(t) = e^{j\tau_0 t} \), \( c_{1u}(\tau) = c_{2u}(\tau) \) may smear the peak at \( \omega_0 \) in the spectrum. Interestingly, without conjugation, the fourth-order cumulant \( c_{4u}(t; \tau_1, \tau_2) = m_{4u}(t; \tau_1, \tau_2, \tau_3) = -4 \delta(\tau_1, \tau_2, \tau_3) \), and hence \( c_{4u}(t; 0, 0, 0) \neq 0 \), which suggests that for QAM signals the test in (6) can be implemented with \( C_{4u}(\alpha; 0, 0, 0) \) instead of \( C_{2u}(\alpha; 0, 0) \).

(s2) Estimating the modulating sequence: \( \mu(t) \)

Because \( \mu(t) \) is deterministic, properties of cumulants [1] and (4) allow us to write for \( k \geq 3 \),

\[
c_{2u}(\tau) = \mu(t) \mu(t + \tau_1) \cdots \mu(t + \tau_k - 1) c_{2u}(\tau),
\]

provided that \( v(t) \) is symmetrically distributed and \( \tau \) is an odd. Setting \( \tau_1 = \cdots = \tau_{k-1} = 0 \) in (7), a closed form is obtained for recovering (within a scalar) the modulating function \( \mu(t) \); e.g., with \( k = 3 \) we find

\[
\mu(t) = \left[ \frac{c_{2u}(t; 0, 0)}{c_{2u}(0, 0)} \right]^{1/3}, \quad c_{2u}(0, 0) \neq 0,
\]

where \( c_{2u}(0, 0) \) is estimated as in (3), and the cycles are available from (s1).

Although (8) is applicable to any (almost) periodic \( \mu(t) \), when \( \mu(t) \) assumes the form in (5), any stationary noise can be theoretically tolerated. To verify this consider

\[
c_{2u}(t; 0) = \begin{cases} 
\sigma_x^2 + \sigma_v^2, & t \in [0, N_1) \ (\text{mod} \ N) \\
\sigma_v^2, & t \in [N_1, N_1 + N_0),
\end{cases}
\]

which indicates that the number of ones \( N_1 \) and zeros \( N_0 \) being observed in a period \( N \) can be readily estimated by inspecting the jumps in \( c_{2u}(t; 0) \) as \( \mu(t) \) changes from 1 to 0. Having \( N_1 \) and \( N_0 \) available, \( \mu(t) \) as well as the "noise floor", \( \sigma_v^2 \), can be estimated.

(s3) Estimating the channel: \( h_0(\tau) \)

Having estimated \( \mu(t) \), it follows from (7) and the definition of cyclic moments and cumulants that the cumulants of the stationary process \( x(t) \) can be estimated using [3], [8]

\[
c_{2u}(\tau) = \frac{C_{2u}(\alpha; \tau)}{M_{2u}(\alpha; \tau)}, \alpha \neq 0 \ (\text{mod} \ 2\pi),
\]

which is an important generalization of the Parzen formula \( k = 2, \alpha = 0 \) [10]. For the missing observations model, the denominator in (10) is nonzero when \( N_1 > (k - 1)N_0 \), [3], [8].
Based on $\hat{c}_k(x)$, known cumulant based linear equations can be solved to estimate $h_0(\tau)$ even when the latter is (non-)causal and (non-)minimum phase (see [7] and references therein). In addition, the following nonlinear least-squares error criterion between $k$th-order sample cumulants computed from the data with periodic misses, and the theoretical cumulants of the assumed MA($q$) process $x(t)$, can be minimized w.r.t. $h_0(h_0(1) \cdots h_0(q))$

$$V_k(h_0) = \sum_{r_1=0}^{r_2-2} M_{k_0}(a; x) [\hat{c}_k(x) - \hat{c}_k(x)]^2$$

(11)

The criterion in (11) is also useful for parametric estimation of ARMA($p,q$) processes and guarantees identifiability when the cumulant lags of [7] are matched in (11). Specifically, it can be shown [8], that with periodic misses and in the presence of any stationary noise, consistency of the ARMA parameter estimates is guaranteed provided that $N_1 \geq \min\{(k-1)N_0, 3p+q\}$ and $k \geq 3$. The latter implies that it is possible to identify ARMA models even when we miss more data than we observe.

2.2 Non-separable case

The sampled signal received through multipath reflections by a mobile radio moving with constant speed can be expressed as in (1), with [12]

$$h(t; \tau) = \sum_{n=1}^{N} H(n; \tau)e^{j\omega_n t}.$$  

(12)

Before addressing the estimation of $h(t;\tau)$, the order $q$ in (1) must be determined. One such method is to check, for the $a$'s specified in Section 2.1, the lag $\tau$ for which

$$C_{11u}(a; \tau) = \frac{1}{T} \sum_{t=0}^{T-1} u^*(t)u(t+\tau)e^{-j\omega_n t}.$$  

(13)

becomes statistically zero. The relevant thresholds can be found using the asymptotic normality and variance expressions for $C_{22u}(a; \tau)$ developed in [4]. Having estimated $q$, we develop cumulant based estimators for $h(t;\tau)$ by integrating ideas from [2] and [12].

First let us assume that $w(t)$ is non-equieprobable asymmetrically distributed symbol stream which implies that $c_{2w}(0,0) \neq 0$. Next we substitute (1) into $c_{2u}(t; \tau)$ and $c_{2u}(t; \tau_1, \tau_2)$ definitions to obtain

$$c_{2u}(t; g, q) = \gamma_{2w} h(t; 0) h(t + q; q),$$  

(14)

$$c_{2u}(t; \tau, q) = \gamma_{2w} h(t; 0) h(t + \tau; \tau) h(t + q; q),$$  

(15)

from which it follows readily, that

$$h(t + \tau; \tau) = \frac{\gamma_{2w}}{\gamma_{2w}} c_{2u}(t; \tau, q).$$  

(16)

In practice $\gamma_{2w}$, $\gamma_{2w}$ are known constants of the input constellation and $\hat{c}_{2u}(t; \tau), \hat{c}_{2u}(t; \tau_1, \tau_2)$ are estimated as in (3) after the cycles $\alpha$ have been determined with the method outlined in Section 2.1. The closed form solution in (16) can be easily extended to higher-order TV cumulants; e.g.,

$$h(t + \tau; \tau) = \frac{\gamma_{2w}}{\gamma_{2w}} c_{2u}(t; 0, \tau, q),$$  

(17)

Because $h(t;0)$, $h(t; q) \neq 0$ $\forall t$, the denominators in (16) and (17) are nonzero. For equiprobable QAM inputs though, $\gamma_{2w} = \gamma_{2w} = 0$, and conjugations are necessary in the cumulant definitions. Let $c_{11u}(t; \tau) = c_{22u}(t; \tau_1, \tau_2, \tau_3)$ and $c_{2u}(t; \tau_1, \tau_2, \tau_3)$

$$m_{22u}(t; \tau_1, \tau_2) - m_{11u}(t; \tau_3)m_{11u}(t + \tau_1; \tau_2 - \tau_1) - m_{11u}(t; \tau_3)m_{11u}(t + \tau_1; \tau_2 - \tau_1)$$  

(16)

[Subscripts indicate the number of conjugated and un-conjugated $u$'s].

Considering once again $q$-slices of second- and fourth-order output cumulants we find

$$h(t; 0)h(t + \tau; \tau) = \frac{\gamma_{2w}}{\gamma_{2w}} c_{2u}(t; 0, \tau, q),$$  

(18)

which allows recovery of $h(t;\tau)$ within a time-varying constant. If the cycles $h(t;0)$ are known, the ambiguity in recovering $h(t;\tau)$ is limited to a complex constant $e^{j\phi}$ with $\phi$ arbitrary [12]. Furthermore, when the cycles in $h(t;\tau)$ are known, identifiability (with the aforementioned ambiguity) is guaranteed only when second- and fourth-order cumulants are employed; i.e., second-order TV cumulants are not sufficient to uniquely determine $h(t;\tau)$ [12].

In special cases simultaneous cycle and TV-parameter estimation is possible by inspecting the TV and cyclic cumulant plots; e.g., if $h(t; \tau) = H(\tau)e^{j\omega_n t}$, then plotting $c_{2u}(t; 0) = \gamma_{2w} \sum_{\rho=0}^{q} h^2(t; \rho)$ reveals both the amplitudes and the cycles present. To assign amplitudes and cycles to the appropriate taps one may try all possible combinations and select the one that matches best (has the smallest least-squared error between the theoretical and) the estimated $\hat{c}_{2u}(t; \tau)$ and/or $\hat{C}_{uu}(t; \tau_1, \tau_2, \tau_3)$.

3 LPTV channel equalization

Once the separable or nonseparable $h(t; \tau)$ has been estimated, the Viterbi algorithm can be employed to recover with minimum probability of error, the information sequence $w(t)$. Alternatively, computational efficiency considerations may dictate the use of a linear MSE equalizer (deconvolver) which must also be LPTV to cope with the channel's time-variation.

Let $e(t) = w(t) - \hat{w}(t)$, $\hat{w}(t) = \sum_{\tau=0}^{Q} d(t; \tau)u(t - \tau)$ be the deconvolved input, and consider as a criterion the time-varying MSE, $\sigma^2(t) = E\{w(t) - \hat{w}(t)^2\}$. For each $t$, minimization of $\sigma^2(t)$ w.r.t. a fixed parameter
\[ d(t; \tau_0), \tau_0 = 0, 1, \ldots, Q \] results in a set of “TV normal equations”

\[
\sum_{\tau=0}^{Q} d(t; \tau) c_{2u}(t; \tau - \tau_0) = \gamma_2 h(t - \tau_0 - \tau_0), \tag{19}
\]

which can be solved to obtain the LPTV-MSE equalizer taps \( d(t; \tau) \). In practice, \( \hat{c}_{2u}(t; \tau) \) is estimated as described in Section 2. Alternatively, if \( h(t; \tau) \) is strictly periodic with period \( N \), synchronized averaging can be employed (see also [6]). Specifically, with \( u(t) = u(t + (j - 1)N) \) denoting stationary sub-processes of the data \( u(t) \), \( t = 0, \ldots, T - 1 \), we may estimate \( c_{2u}(t; \tau) \) using

\[
\hat{c}_{2u}(t; \tau) = \frac{1}{L} \sum_{l=1}^{L} u(l) u(l + \tau), \quad L \equiv \lfloor T/N \rfloor. \tag{20}
\]

In fact, it follows that the sampled version of (19) can be reached if one minimizes the time-averaged MSE

\[
J_T = \frac{1}{T} \sum_{t=0}^{T-1} \sum_{n=0}^{N-1} \sum_{l=1}^{L} \left[ e(t) - \sum_{m=0}^{M} d(l; \tau) u_l(t - \tau) \right]^2,
\]

which can be expressed as \( J_T = L^{-1} \sum_{l=1}^{L} J_T(l) \), where \( J_T(l) \equiv N^{-1} \sum_{n=0}^{N-1} \sum_{m=0}^{M} c_{2u}^2(l) \). For each \( l \), minimizing \( J_T(l) \) w.r.t. \( d(l; \tau) \), since \( c_2(t) = w(t) - \sum_{\tau=0}^{\infty} d(\tau) u_l(t - \tau) \). Therefore, when \( h(t; \tau) \) is strictly periodic, the LPTV equalizer \( d(t; \tau) \) is equivalent to a bank of \( N \) linear time-invariant equalizers \( d(l; \tau) \), each operating on the \( N \) separate sub-processes \( u_l(t) \) (see also Fig. 2). This interpretation allows application of on-line line algorithms such as the Recursive Least-Squares for the solution of (19).

### 4 Simulations

To illustrate the cycle estimation of (6) we modulated an MA(3) process with coefficients \([1, -0.75, 0.35, 0.43] \) by \( \mu(t) \) of (5) with \( N_1 = 4 \), \( N_0 = 6 \). Fig. 3a shows that \( |c_{2u}(\alpha; 0)| \), estimated with \( T = 512 \) data, peaks at \( \omega_0 = 0.2\pi \) verifying correctly a period of \( N = 2\pi/\omega_0 = 10 \). Next, we used \( \mu(t) = \sin(t) \), added 10dB Gaussian noise, and obtained the \( |c_{2u}(\alpha; 0)| \) plot of Fig. 3b which shows peaks at \( \alpha = 0 \) (due to the AGN) and \( \alpha = \pm 2\pi \).

To test the \( \mu(t) \) estimation we modulated the same \( \text{MA}(3) \) process by \( \mu(t) = 1 + \sin(2\pi t/40 + 0.1\pi) \) and generated \( T = 4,096 \) noisy data \( \sigma_v^2 = 1.37, \sigma_v^2 = 0.28 \). We used synchronized averaging to compute \( c_{2u}(t; 0) \) from which we found \( \sigma_v^2 = 0.22 \) and recovered \( \sigma_v^2 \mu(t) \) depicted in Fig. 4a. From this figure one may also infer the phase shift since \( \phi = (10 - 8) \times 2\pi/40 = 0.1\pi \). We also tested the \( \mu(t) \) of (5) with \( N_1 = 6, N_0 = 4 \) and \( \sigma_v^2 = 0.11 \). Fig. 4b shows \( c_{2u}(t; 0) \) computed with \( T = 2,048 \) which drops at \( t = 6 \) and indicates correctly \( N_1 = 6, N_0 = 4 \) and \( \sigma_v^2 = 0.10 \).

To demonstrate the channel estimation step we simulated a mixed-phase channel with taps \([1, 0.75, 0.4, -1.34] \) which we modulated by \( \mu(t) \) of (5) with \( N_1 = 4, N_0 = 6; T = 1,024 \) noisy data at 0dB were used in the criterion (11) with \( k = 3 \). The resulting log-magnitude and phase plots of Figs. 5a and 5b (solid: true, dashed: mean estimates, dotted: st. dev. bounds from 50 runs) corroborate the ability of cumulants to estimate non-minimum phase models even when more misses than observations are present. Note that with \( k = 2 \) in (11) one captures only the minimum phase (dash-dotted) curve of Fig. 5b.

To check the non-separable case we simulated an LPTV channel with \( q \geq 2 \) and taps: \( h(t; 0) = e^{-j0.4t} \) \( h(t; 1) = (0.6 + j0.8) e^{j0.8t} \), \( h(t; 2) = (0.8 + j0.6) e^{j0.8t} \) and \( \sigma_v^2 = 0.8, T = 10,240 \). From Figs. 6a and 6b which depict \( c_{2u}(\alpha; 0), \hat{c}_{2u}(\alpha; 0), \hat{c}_{2u}(\alpha; 0) \), and \( \hat{c}_{2u}(\alpha; 1) \) respectively, we obtain by inspection \( \sigma_v^2 = 0.81 \), and the coefficient estimates: \( (0.58 + j0.79)e^{j\pi/3} \), \( (0.95 - j0.003)e^{j0.4t} \), and \( (0.83 + j0.59)e^{j0.4t} \). To assign these coefficients to the correct taps we matched each of the 3! combinations to the estimated \( c_{2u}(t; 2) \). Note also that in Fig. 6b, the AGN component is suppressed and the tap-cycles appear at \( 3\omega_0 \), \( \omega_0 = 1, 2, \ldots \). Finally, to verify the LPTV-MSE equalizer we generated randomly spaced (Bernoulli) spikes with occurrence rate 0.3 and amplitude controlled by an \( N(0, 1) \) random variable. This i.i.d. input was passed through an MA(3) channel with coefficients \([0.198, -0.034, -0.035] \) and the output was modulated by \( \mu(t) = |\sin(\pi/5 + \pi/5)| \), and corrupted by 30dB colored AGN. Figs. 7a and 7b show \( u(t) \) (solid lines) vs. \( u(t) \) (dotes) from a 5-channel equalizer and a single MSE equalizer output, respectively.

### References

Figure 1: LPTV channel estimation setup

Figure 2: LPTV channel equalizer

Figure 3: Estimating the cyclic frequencies

Figure 4: Estimating the modulating sequence

Figure 5: Estimating the channel w/ misses

Figure 6: Non-separable LPTV channel

Figure 7: LPTV- vs. LTI-MSE equalizers


