Adaptive RRQR-based Factorization: Improving the Algorithm Tracking Capabilities

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Abstract

We recently proposed an updating scheme [1] based on the Rank-Revealing QR (RRQR) factorization described by Chan [2]. The RRQR algorithm provides an attractive alternative to SVD-based or eigen-based techniques for subspace selection. In addition, the updating RRQR-based technique, used to track time-varying information, takes advantage of the simplicity of the regular QR updating scheme. In this paper we propose a procedure to improve the performance of the RRQR-based updating scheme. The refinement uses the signal subspace information and takes advantage of the Hermitian property of the signal correlation matrix. It is implicitly equivalent to applying one subspace iteration step to the estimated signal subspace. Simulations show the performance obtained is similar to that obtained using classical eigen-based techniques.

1. Introduction

Subspace decomposition techniques are powerful tools used in areas of signal processing in which the signal information is usually obtained via eigen-based techniques. These techniques are numerically stable but computationally intensive methods to update the time-varying information. As a result, various alternatives have been proposed to reduce the computational load associated with the signal information estimation [3-6]. We recently proposed an updating scheme [1] based on the Rank-Revealing QR (RRQR) factorization described by Chan [2] and applied it to the Direction-Of-Arrival (DOA) problem. This technique allows for tracking of moving sources by taking advantage of the simplicity of the regular QR updating scheme and the rank-revealing property of the RRQR factorization. In this paper, we propose a refinement to the original RRQR-based tracking algorithm. This refinement is implicitly equivalent to applying one subspace iteration step to the original RRQR-based signal subspace estimate. Section 2 reviews subspace methods and the RRQR-based tracking procedure. Section 3 answers questions raised in our earlier correspondence concerning pivoting issues [1]. Section 4 develops the refinement scheme. Section 5 presents experimental results. Conclusions are presented in Section 6.

2. The Rank-Revealing QR Factorization applied to tracking of time-varying information

Let us first review how eigen-based techniques are used to estimate signal information. Consider a linear equi-spaced array of N sensors receiving p narrowband signals. Under the assumption of non-dispersive propagation, sensors without distortion, and envelope variations that are slow relative to the carrier frequencies of the narrowband signals, the signal received by the passive array is:

\[ r(t) = iv(e^{\lambda t}) + n(t). \]

The signal \( s(t) \) represents the narrowband signal, the mode matrix \( M(\theta) \) is defined as:

\[ M(\theta) = [m(\theta_1), ..., m(\theta_p)], \]

where \( m(\theta) \) represents the delay with which the signal impinges on each of the sensors, and \( n(t) \) is additive noise. Under the assumption that the zero-mean noise is uncorrelated with the signals, the theoretical spatial correlation for the received signal is given by:

\[ R = E[x(t)x(t)^H] = MM^H + \sigma^2 I, \]

where \( S = E[s(t)s(t)^H] \) is the signal correlation matrix, and \( \sigma^2 \) is the noise variance. The estimated correlation
matrix obtained using a samples of the sensor output 
x(t) is given by:

\[ R = \frac{1}{n} \sum_{i=1}^{n} x(t)x^H(t) = \frac{1}{n} X^H X, \]

where \( X = [x(1), \ldots, x(n)]^H \). Numerous eigen-based and SVD-based techniques have been proposed to identify the DOA information from the signal or noise subspace obtained from \( R \) [8]. When the information is time-varying (i.e., in the DOA setting when the sources are moving), a continuous updating of the signal or noise subspace information is required to track the moving sources. SVD or Eigendecomposition (EVD) techniques are expensive to update and "approximations" to the theoretical matrix \( R \) have been proposed in an effort to reduce the computations associated with the updating procedure [4,12]. The RRQR-based method proposed earlier [1] takes a different approach as it is based on the QR factorization of the noise-free (signal) correlation matrix \( R = R - \sigma^2 I = M S M^H \). Note that the theoretical matrix \( R \) is rank deficient when \( p < N \). Thus, applying the QR algorithm with column pivoting to \( R \) leads to:

\[ R_{\Pi} = Q U = \begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}, \]

where \( Q \) is orthonormal, \( U_{\Pi} \) is a permutation matrix, \( U_{11} \) is \( p \times p \), \( U_{22} \) is \((N-p) \times (N-p) \), and \( U_{12} \) is \((N-p) \times p \). Recall that in the classical QR factorization with column pivoting the permutation matrix \( U_{\Pi} \) is chosen so that at each iteration the column chosen for pivot is the one farthest away from the subspace spanned by the columns already selected [9]. When \( R_{\Pi} \) has rank \( p \), then \( U_{22} = 0 \), and \( \text{Range}(R_{\Pi}) = \text{Range}(U_{11}) \). Thus, theoretically the DOA information can be obtained from a QR factorization of the noise-free correlation matrix \( R_{\Pi} \). However, when the received signal and noise correlation matrix \( R \) is estimated, \( R_{\Pi} \) becomes nearly rank-deficient only. In such a case the QR factorization is not reliable, as it may produce a block \( U_{22} \) which is not small in norm. The problem can be avoided with the RRQR factorization introduced by Chan; this procedure uses a pivoting strategy designed to produce a block \( U_{22} \) small in norm when the matrix to be factorized is nearly rank-deficient [1,2,7].

Tracking moving sources requires the correlation matrix to be updated and the signal/noise subspaces to be continuously recomputed. A fixed-length moving window is used in the following adaptive implementation. The time-varying estimated correlation matrix \( R(k+1) \), obtained by adding a \((k+1)\)th snapshot \( x(k+1) \) and deleting the old snapshot \( x(1) \) to \( R(k) \) is given by:

\[ R(k+1) = \frac{1}{n} [X^H X + x(k+1)x^H(k+1) - x(1)x^H(1)], \]

where \( X = [x(1), \ldots, x(n)]^H \).

The adaptive implementation of the RRQR factorization presented in [1] is a two-step procedure. At each iteration, we first apply the basic QR updating scheme to \( R \). Next, we apply the RRQR restricted pivoting (i.e., column permutations) scheme to reveal the numerical rank of \( R \) and to obtain signal and noise subspace estimates. Simulations showed that the basic RRQR-based updating scheme can be used to estimate signal information for medium to high SNR ratios [1].

3. Pivoting Issues in the RRQR-Based Adaptive Scheme

Recall that the RRQR-based adaptive scheme does not involve pivoting before applying the column permutations scheme proposed by Chan. This is done to take advantage of the simplicity of the rank-1 QR updating procedure [1,9]. By comparison, Chan's original method uses an initial QR decomposition with pivoting in the first step of the algorithm [1,2]. We investigated the effect of this initial pivoting on the "quality" of the estimated subspaces obtained with the RRQR factorization of the ill-conditioned matrix \( R \). The columns labelled 'Chan's approach' in Table 1 present the average distance (for 1000 trials) between the signal subspace generated using the RRQR updating technique and that obtained using the EVD of \( R \) for various SNRs. Note that in Table 1 "with (without) pivoting" means that a QR factorization with (without) pivoting is used in the first step of the RRQR factorization. Table 1 shows that no significant difference in the estimated subspaces is found when using QR factorization with or without pivoting in step one of the RRQR algorithm.

Recall that the second step of the RRQR factorization involves a succession of specific column permutations which reveal the potential near-rank deficiency of the matrix under study. Prasad and Chandna [3] presented an application of the RRQR factorization to the DOA problem for fixed source locations. Specifically, they proposed the use of fewer column permutations than originally used by Chan, and to iterate from \( N(p) \) to \( N(p+1) \) instead of from \( N \) to \( p+1 \). The columns labelled 'Prasad's approach' in Table 1 present the average distance (for 1000 trials) between the signal subspace generated using Prasad's RRQR updating technique and the signal subspace.
obtained from using the EVD of \( R \). In addition, the influence of the pivoting in step one of the RRQR factorization is investigated. Table 1 shows that poorer subspace estimates are obtained when reducing the number of column permutations. This could be expected as fewer column permutations are allowed to reveal the rank structure of the matrix under study, leading to poorer subspace estimates.

Furthermore, Table 1 shows that a distinct worsening of the subspace estimates is obtained when no pivoting is used in step one of Prasad's RRQR factorization. This can be expected as the basic QR factorization is likely to be further away from the RRQR factorization structure when no column pivoting is allowed in the first step of the RRQR algorithm and fewer column permutations are used to reveal the rank structure.

4. RRQR-Based Factorization Refinement

Simulations showed that the RRQR-based adaptive scheme could be used to estimate signal and noise subspace information for medium to high SNRs. The purpose of this section is to show that a better signal subspace estimate may be obtained by taking advantage of the Hermitian property of the noise-free correlation matrix \( R \). Recall that the RRQR-based algorithm leads to the following decomposition: \( R = Q M \), where \( Q \), \( U \) and \( \Pi \) are defined in Section 2. Using the fact that \( R \) is hermitian leads to:

\[
R = Q U \Pi \Pi^T = R \Pi^H = \Pi U^H Q^H.
\]

Next, recall that the signal and noise subspaces \( Q_s \) and \( Q_n \) are contained in the matrix \( Q \). Thus,

\[
R Q_s = R \Pi^H U^H Q_s = \Pi U^H \left[ \begin{array}{c} 0 \\ Q_n^H \\ Q_s^H \end{array} \right] = \Pi \left[ \begin{array}{c} 0 \\ U_{11}^H \\ U_{22}^H \end{array} \right] = \Pi U_{11}^H + \Pi U_{22}^H.
\]

and therefore,

\[
R Q_s = \left[ \begin{array}{c} \Pi U_{11}^H \\ \Pi U_{12}^H \\ \Pi U_{22}^H \end{array} \right] = \Pi U_{11}^H + \Pi U_{12}^H + \Pi U_{22}^H.
\]

The matrix \( \hat{Q}_s = R Q_s \) may be viewed as the result of applying one subspace iteration step to the estimated signal subspace \( Q_s \) [10]. Note that no additional computation is needed to obtain the new signal subspace \( \hat{Q}_s \) as \( \Pi \) and \( U \) have been computed already. This "implicit" iteration scheme improves the results obtained, as shown in the next section. The drawback is that the resulting iterated signal subspace is no longer orthonormal. Thus, an additional orthonormalization step may have to be applied to \( \hat{Q}_s \), depending on the DOA estimator used. Such an orthonormalization step is needed when using the Minimum-Norm algorithm [11], for example. However, no reorthonormalization step is needed when using the MUSIC estimator [8].

5. Simulation Results

We now study the performance of the RRQR-based adaptive algorithm by comparing the signal subspace obtained when updating the noise-free correlation matrix \( R \), with the one obtained with the EVD of \( R \). The largest principle angle between estimated subspaces is used for comparison [9]. Next, the Min-Norm algorithm is chosen to track the source locations in 3 cases: RRQR factorization with and without a subspace iteration step; and the EVD algorithm.

We consider the case of two sources impinging on a ten-element array. The first source is assumed to be fixed at a normalized angle \( \theta_1 = 40^\circ \), the second source location \( \theta_2 \) is linear time-varying. 100 snapshots are chosen to estimate the correlation matrix, and 200 updates are used for tracking. The two sources are assumed to have a SNR equal to 5dB, 10dB, and 15dB successively. Table 2 presents the average distance between the signal subspace generated using the RRQR updating technique (with and without refinement) and the signal subspace obtained from the EVD of \( R \). This table shows that the implicit one-step subspace iteration improves the signal subspace estimates. Next, Fig. 1 presents the estimated linear time-varying source location \( \theta_2 \) for various SNRs, obtained using the Min-Norm algorithm. Fig. 1 shows that the performance of the algorithm improves greatly when using the implicit subspace iteration step, as the RRQR-with-refinement DOA angle is very close to obtained with the EVD.

Finally, we investigate the robustness of the RRQR-based adaptive scheme to noise power errors. We use the same array parameters as those defined in Fig. 1, and SNR equal to 10dB. Fig. 2 presents the average distance and \( \pm 1 \) standard deviation confidence intervals, between the signal subspaces generated using the RRQR-based (with/without refinement) and the signal subspaces obtained from the EVD of \( R \). Fig. 2 shows that the "refined" RRQR-based technique is quite robust to noise power errors.

6. Conclusions

We have presented an improvement to a Rank-Revealing QR updating procedure proposed earlier. The method takes advantage of the simplicity of the QR
updating scheme, the rank-revealing property proposed by Chan, and the Hermitian property of correlation matrices. In addition, we have compared our results with the approximation proposed by Prasad et al. Simulations show that the "refined" RRQR-based updating scheme approximate signal (and noise) subspaces closely and is robust to noise power errors.

References:


Figure 1: Lin. time-varying source $\theta$, location (degrees) for RRQR-based and Eigen-based signal subspaces ($\theta = 40^\circ$, SNR = 5, 10, 15 dB)
Table 1: Distance (in degrees) between RRQR-based and Eigen-based signal subspaces, with (without) pivoting used in the QR step of RRQR algorithm ($\theta_1=30^\circ$, $\theta_2=32^\circ$, 10 sensors, 1000 trials).

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>-10</th>
<th>0</th>
<th>10</th>
<th>-10</th>
<th>0</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Chan's approach mean (standard dev.)</td>
<td>Prasad's approach mean (standard dev.)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>QR w/ pivoting</td>
<td>46.80 (13.65)</td>
<td>17.53 (5.71)</td>
<td>2.24 (0.69)</td>
<td>47.05 (14.11)</td>
<td>26.95 (12.55)</td>
<td>4.85 (4.09)</td>
</tr>
<tr>
<td>QR w/o pivoting</td>
<td>46.76 (13.58)</td>
<td>17.45 (5.70)</td>
<td>2.21 (0.63)</td>
<td>59.59 (13.79)</td>
<td>50.81 (23.62)</td>
<td>14.47 (8.72)</td>
</tr>
</tbody>
</table>

Table 2: Distance between signal subspaces obtained using the RRQR alg. with (w/o) refinement and the EVD, DOA angles for the linear time-varying source $\theta_1$ (various SNRs, $\theta_1=40^\circ$, 200 updates).

<table>
<thead>
<tr>
<th>SNR (dB)</th>
<th>5</th>
<th>10</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mean, standard deviation (degrees)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Signal Subspaces distance, RRQR w/o refinement</td>
<td>3.4430, 1.1514</td>
<td>1.0206, 0.2937</td>
<td>0.3238, 0.0883</td>
</tr>
<tr>
<td>DOA angles difference for $\theta_2$, RRQR w/o refinement</td>
<td>0.7445, 0.5339</td>
<td>0.2119, 0.15012</td>
<td>0.0452, 0.0288</td>
</tr>
<tr>
<td>Signal Subspaces distance, RRQR with refinement</td>
<td>0.2186, 0.1133</td>
<td>0.0206, 0.0096</td>
<td>0.0021, 0.0009</td>
</tr>
<tr>
<td>DOA angles difference for $\theta_2$, RRQR with refinement</td>
<td>0.0200, 0.0210</td>
<td>0.0013, 0.0011</td>
<td>0.0002, 0.0002</td>
</tr>
</tbody>
</table>

Figure 2: Avg distance (deg) betw. Eigen-based and RRQR-based signal subspaces as a function of errors in the SNR estimation (true SNR=10 dB, $\theta_1=40^\circ$, $\theta_2$ lin. time-varying, 200 upd.).