The Design of Structurally Constrained Minimum Mean-Square Error Block Matrix Filters

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Abstract

The design of minimum mean-square error (MSE) block matrix filters subject to structural constraints is presented. A general technique of imposing the structural constraints directly in the MSE formulation is proposed and applied to diagonal, circulant, non-causal Toeplitz and causal Toeplitz matrix filters. Simulations are submitted and discussed.

1 Introduction

Block matrix filters are often required to process signals having certain properties that in turn constrain the matrix structures. For example, a time-invariant $N \times N$ linear discrete-time block matrix filter system results in Toeplitz structures [4, pp. 13-15] while output due to circular convolution [4, p. 951] is.

It is of general interest, then, to find a structurally constrained block matrix filter $h$ that minimizes a prescribed type of error between the input and the output, namely, the mean-square error (MSE). The main advantage of this approach is that the resulting formulation produces a quadratic (and hence convex) function in the matrix filter $h$, implying that a minimum MSE solution will yield a global minimum [1]. In addition, the result provides a closed-form solution.

In the following derivations, the expectation operator $E\{}$ is assumed to be implicit in the equations, the main focus being the development of the general formulation. We further recall that the expectation of a matrix (or vector) is the expectation of each of its elements, that is,

$$E\left\{ \sum_i \sum_j r_i s_j \right\} = \sum_i \sum_j E\left\{ r_i s_j \right\}$$

(1.1)

for $r_i, s_j \in \mathbb{R}$ and $i, j \in \mathbb{Z}$, so that application of the expectation is straightforward in the matrix formulations.

The input $x$ is a discrete sequence of real-valued samples $x(n)$ and bounded. It is composed of a signal sequence $s(n)$ and noise (or non-signal) sequence $n(n)$, described by an operation $\otimes$, such that

$$x(n) = s(n) \otimes n(n)$$

(1.2)

This allows for homomorphic filtering [6].

2 General Minimum MSE Formulation

We consider the minimization of the error

$$e^T e = (y - d)^T (y - d) = \|y - d\|^2$$

(2.1)

where $e \in \mathbb{R}^N$ is the column vector representing the error between the actual output vector $y \in \mathbb{R}^N$ and the desired output vector $d \in \mathbb{R}^N$, $^T$ is the transpose, and $\| \cdot \|_2$ denotes the Euclidean norm. The output is

$$y = hx$$

(2.2)

where $x \in \mathbb{R}^N$ is the input column vector and $h \in \mathbb{R}^{N \times N}$ is the filter matrix.

The minimization of (2.1) is achieved through

$$\nabla_h e^T e = \nabla_h (x^T h^T d) - \nabla_h (d^T h x) + \nabla_d (d^T d) = 0$$

(2.3)

where $\nabla_h (P)$ is the gradient of $P$ with respect to $h$ [3]. This is our minimum MSE formulation.

In (2.3) we note that $\nabla_h (d^T d) = 0$ since the desired vector is not a function of the matrix $h$. The other parts of the equation are functions of $h$, however, and we are thus led to establishing the properties of (2.3).

PROPERTY 2.1 For any finite vectors $d, x \in \mathbb{R}^N$ and any finite matrix $h \in \mathbb{R}^{N \times N},$

$$\nabla_h (x^T h^T d) = \nabla_h (d^T h x)$$

(2.4a)

Proof. This is a well-known result of real matrix theory and is easily verified. The main result is

$$\frac{\partial}{\partial h_{p,q}} (x^T h^T d) = x_p d_q$$

(2.4b)

PROPERTY 2.2 For any finite vector $x \in \mathbb{R}^N$ and any finite matrix $h \in \mathbb{R}^{N \times N},$

$$\frac{\partial}{\partial h_{p,q}} (x^T h^T h x) = \sum_{i=0}^{N-1} h_{p,i} x_i x_q + \sum_{j=0}^{N-1} h_{p,j} x_j x_q + 2h_{p,q} x_q^2$$

(2.5)

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Proof. Let the matrix \( g = h^T h \) and vector \( y = g x \) so that each element of \( y \) is
\[
y_i = \sum_{j=0}^{N-1} g_{ij} x_j = \sum_{k=0}^{N-1} \left( \sum_{i,k} h_{k,i} h_{k,j} \right) x_j \tag{2.6}
\]
as \( h_{i,k} = h_{k,i} \) for the transpose of \( h \). Furthermore, let the scalar \( P = x^T y \), or
\[
P = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} x_i x_j \left( \sum_{k=0}^{N-1} h_{k,i} h_{k,j} \right) \tag{2.7}
\]
We can take the gradient of \( P \) with respect to \( h_{pq} \) as
\[
\frac{\partial P}{\partial h_{pq}} = \frac{\partial}{\partial h_{pq}} \left( \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} x_i x_j h_{p,i} h_{j,p} \right) = \sum_{i=0}^{N-1} h_{p,i} x_i x_j + \sum_{j=0}^{N-1} h_{p,j} x_j x_q + 2 h_{p,q} x_i^2 \tag{2.5}
\]
since the summation is taken over the values where the partial derivatives have a nonzero value.

After some manipulation and including the above properties, \( (2.3) \) now becomes
\[
E \left\{ \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} h_{p,i} x_i x_j + \sum_{j=0}^{N-1} h_{p,j} x_j x_q + 2 h_{p,q} x_i^2 \right\} = E \left\{ 2 x_i d_i \right\} \tag{2.8}
\]
Eq. (2.8) is the general formulation used to derive the minimum MSE formulation for a matrix with any given structure.

3 Diagonal Matrix Filters

We begin with the structurally simplest of the constrained matrix filters, the time-varying memoryless matrix filter. We impose the gradient given by \( (2.3) \) with a diagonal \( h \). For the scalar \( P = d^T h x \),
\[
\frac{\partial P}{\partial h_{ii}} = E \{ d_i x_i \} \tag{3.1}
\]
Similarly, for the scalar \( P = x^T h^T h x \), we have
\[
\frac{\partial P}{\partial h_{ii}} = E \{ h_{ii} x_i x_i \} = h_{ii} E \{ x_i x_i \} \tag{3.2}
\]
Equating \( (3.2) \) with \( (3.1) \) we solve for each \( h_{ii} \) as
\[
h_{ii} = \frac{E \{ d_i x_i \}}{E \{ x_i x_i \}} \quad i = 0, 1, \ldots, N - 1 \tag{3.3}
\]
which is expected, and \( h_{i,j} = 0 \) for \( i \neq j \).

4 Circulant Matrix Filters

For circulant matrices, \( h_{ij} = h_{m_N(i+n_0),m_N(j+n_0)} \) for any \( i, j, n_0 \in Z \), and where \( m_N(k) = \text{mod}_N(k) \), the modulo operator. Given a scalar \( P = d^T h x \), it can be shown that
\[
\frac{\partial P}{\partial h_{i,0}} = \frac{\partial P}{\partial h_{1,1}} = \cdots = \frac{\partial P}{\partial h_{N-1,N-1}} = \sum_{i=0}^{N-1} d_i x_i \tag{4.1}
\]
\[
\frac{\partial P}{\partial h_{0,1}} = \frac{\partial P}{\partial h_{1,2}} = \cdots = \frac{\partial P}{\partial h_{N-1,0}} = \sum_{i=0}^{N-1} d_i x_m(i+i+1)
\]
\[
\vdots
\]

\[
\nabla_h (d^T h x) = \nabla_h (x^T h^T h x) =
\]

\[
\left( \begin{array}{c}
\sum_{i=0}^{N-1} d_i x_i \\
\sum_{i=0}^{N-1} d_i x_m(i+i+1) \\
\vdots \\
\sum_{i=0}^{N-1} d_i x_m(i+i+N-1)
\end{array} \right)
\]

with circulant structure.

For the scalar \( P = x^T h^T h x \), it can be shown that the column vector of partial derivatives can be represented by the linear system of equations
\[
I_0 = \left[ \frac{\partial P}{\partial h_{0,0}}, \frac{\partial P}{\partial h_{0,1}}, \ldots, \frac{\partial P}{\partial h_{0,N-1}} \right]^T = 2A_C h_C \tag{4.3}
\]
where
\[
A_C =
\left( \begin{array}{cccc}
\sum_{i=0}^{N-1} x_i x_i & \ldots & \sum_{i=0}^{N-1} x_i x_m(i+i+1) \\
\sum_{i=0}^{N-1} x_i x_m(i+i+1) & \ldots & \sum_{i=0}^{N-1} x_i x_m(i+i+N-1) \\
\vdots & \ddots & \vdots \\
\sum_{i=0}^{N-1} x_i x_m(i+i+N-1) & \ldots & \sum_{i=0}^{N-1} x_i x_i
\end{array} \right)
\]
is the \( N \times N \) circulant system coefficient matrix, and
\[
h_C = (h_{0,0}, h_{0,1}, \ldots, h_{0,N-1})^T \tag{4.5}
\]
is the \( N \times 1 \) real coefficient column vector.
It can be shown that the generalized gradient
\( \nabla_h(x^T h^T x) = (h_0(0), h_0(N-1), \ldots, h_0(1)) \) where
\( h_0(N-m) \) is a version of \( h_0 \) shifted \( m \) times to the
right. Due to circulant symmetry we solve for only one
row because each row is just a shifted version of any
other row. Therefore, we can solve the linear system of
equations
\[
A_C h_C = p
\] (4.6)
where \( p \) is the \( N \times 1 \) real column vector from the first
row of the matrix \( \nabla_h(d^T h x) \), that is,
\[
p = \left( \sum_{i=0}^{N-1} d_i x_i, \sum_{i=0}^{N-1} d_i x_{m(1+i)}, \ldots, \sum_{i=0}^{N-1} d_i x_{m(N+N-1)} \right)^T
\] (4.7)
and \( A_C \) is given by (4.4) while \( h_C \) is given by (4.5).
The solution to the coefficients \( h_{ij} \) for the circulant
matrix are obtained from
\[
h_C = A_C^{-1} p \] (4.8)

5 Non-Causal Toeplitz Matrix Filters

For Toeplitz matrices, \( h_{ij} = h_{i+j+\tau} \), \( i,j,\tau \in Z \). Given a scalar \( P = d^T h x \), it can be shown that
\[
\frac{\partial P}{\partial h_{0,0}} = \frac{\partial P}{\partial h_{1,1}} = \ldots = \frac{\partial P}{\partial h_{N-1,N-1}} = \sum_{i=0}^{N-1} d_i x_i
\]
\[
\frac{\partial P}{\partial h_{1,0}} = \frac{\partial P}{\partial h_{2,1}} = \ldots = \frac{\partial P}{\partial h_{N-1,N-2}} = \sum_{i=0}^{N-2} d_{i+1} x_i
\]
\[
\frac{\partial P}{\partial h_{0,1}} = \frac{\partial P}{\partial h_{1,2}} = \ldots = \frac{\partial P}{\partial h_{N-2,N-1}} = \sum_{i=0}^{N-2} d_{i+1} x_{i+1}
\]
\[\vdots\]
Therefore, the resulting matrix is a Toeplitz matrix
where the elements of the diagonals are summed, viz.
\[
\nabla_h(d^T h x) = \nabla_h(x^T h^T d) = \left( \begin{array}{cccccc}
\sum_{i=0}^{N-1} d_i x_i & \sum_{i=0}^{N-2} d_i x_{i+1} & \cdots & \sum_{i=0}^{0} d_i x_{i+N-1} \\
\sum_{i=0}^{N-2} d_{i+1} x_i & \sum_{i=0}^{N-1} d_i x_i & \cdots & \sum_{i=0}^{N-2} d_i x_{i+1} \\
\sum_{i=0}^{0} d_{i+N-1} x_i & \cdots & \sum_{i=0}^{N-1} d_i x_i
\end{array} \right)
\] (5.1b)

Let
\[
y_i = \sum_{j=0}^{N-1} h_{ij} x_j \] (5.2)

Then it can be shown that
\[
\nabla_h(x^T h^T x) = \left( \begin{array}{cccccc}
2 \sum_{i=0}^{N-1} y_i x_i & 2 \sum_{i=0}^{N-2} y_i x_{i+1} & \cdots & 2 \sum_{i=0}^{0} y_i x_{i+N-1} \\
2 \sum_{i=0}^{N-1} y_i x_{i-1} & 2 \sum_{i=0}^{N-2} y_i x_{i} & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
2 \sum_{i=0}^{0} y_i x_{i-N-1+N} & \cdots & \cdots & 2 \sum_{i=0}^{N-1} y_i x_i
\end{array} \right)
\] (5.3)

Equating (5.3) with (5.1b) through (2.5) we can
solve for the \( 2N-1 \) coefficients \( h_{ij} \) through the system of
the general form
\[
A_T h_T = p
\] (5.4)
where \( p \) is the \( (2N-1) \times 1 \) solution vector

Let
\[
h_T = \left( h_{0,0}, h_{0,1}, \ldots, h_{0,N-1}, h_{1,0}, h_{1,1}, \ldots, h_{1,N-2}, \ldots, h_{N-1,0} \right)^T
\] (5.5)

where the first \( N \) entries are the solutions to the system
along the rows, and the following \( N-1 \) entries are the solutions to the system along the columns which
are separated by a vertical line in the vector in (5.5),
and where \( h_T \) is the \( (2N-1) \times 1 \) column vector con-
taining the \( h_{ij} \) arranged as follows:

The system coefficient matrix has the following
structure:
\[
A_T = \left( \begin{array}{cc}
B_{N \times N} & \vdots \\
\vdots & \vdots \\
\vdots & \vdots \\
M_{(N-1) \times N} & Q_{N \times (N-1)}
\end{array} \right)
\] (5.7)

where each block has the structure given below.
The solution to the coefficients \( h_{ij} \) for the Toeplitz
matrix are obtained from
\[
h_T = A_T^{-1} p
\] (5.8)
6 Causal Toeplitz Matrix Filters

A causal Toeplitz matrix is lower-triangular and the partial derivatives of the elements are zero for $j > i$. We follow the same procedure as that for deriving the original Toeplitz matrix, noting that we need to solve only along the columns. Essentially we get the same result as in (5.4), solving only for

$$h_{T,\text{causal}} = (h_{0,0}, h_{1,0}, \ldots, h_{N-1,0})^T$$

(6.1)

In solving for the system given by (5.4) but with causality imposed, $p$ is the $N \times 1$ solution vector

$$p = \left( \sum_{i=0}^{N-1} d_i z_i, \sum_{i=0}^{N-2} d_{i+1} z_i, \ldots, \sum_{i=0}^{0} d_{N-1} z_i \right)^T$$

(6.2)

and $A_{T,\text{causal}}$ consists of the first row and column of $B_{N \times N}$ and the rest is composed of the matrix $R_{(N-1)\times(N-1)}$.

7 Simulation and Analysis

The filter system was simulated on a generic SISO matrix filter shown in Fig. 7.1. The signal $s = e^{-t}$ is shown in Fig. 7.2(a). The total input $x = s + n$ assumes additive white noise with SNR=10dB and is shown in Fig. 7.2(b). The signals consist of 32 bits of data. The estimates of the correlation matrices are the actual correlation matrices except for the input autocorrelation where the noise mean energy was equally distributed along the main diagonal of the signal autocorrelation matrix in order to preserve the assumption that the signal and noise are uncorrelated. For estimation $d = s$. For detection $d = U_0 = (1, 0, \ldots, 0)^T$.

Figures 7.3, 7.4, and 7.5 show (a) the estimation matrix, (b) the estimation output, and (c) the detection output for diagonal, circulant, and causal Toeplitz matrix filters, respectively. For diagonal systems, the estimation MSE=.2704 and the detection MSE=.2312. For circulant systems, the estimation MSE=.0276 and the detection MSE=.0479, but note the periodic extension at the tail end of the outputs. For causal Toeplitz systems, the estimation MSE=.0226 and the detection MSE=.0202.

Diagonal systems are the poorest overall performers since they work with only $N$ coefficients. Circulant systems force periodicity in estimation and detection applications, so they are best suited for periodic inputs. Toeplitz systems are time-invariant non-periodic and do not show periodic extensions as in circulant filters. The low MSE results from the constrained optimization approach presented here.

8 Conclusion

The design of structurally constrained minimum MSE block matrix filters was presented. A general formulation for deriving the minimum MSE matrix filters with any given structure was proposed and applied to diagonal, circulant, non-causal Toeplitz and causal Toeplitz matrix filters. Simulations showed good correspondence between the theory and applications.

References


Figure 7.1 SISO matrix filter [2].

Figure 7.2 (a) Signal $s = e^{-t}$, (b) $x = s + n$.

Figure 7.3 Diagonal system: (a) Estimation matrix, (b) estimation output, (c) detection output.

Figure 7.4 Circulant system: (a) Estimation matrix, (b) estimation output, (c) detection output.

Figure 7.5 Causal Toeplitz system: (a) Estimation matrix, (b) estimation output, (c) detection output.