Wavelets in Optimal Radar Range-Doppler Imaging

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Abstract

It is shown that the most accurate reconstruction of a range-Doppler target density that can be computed from range-Doppler target density that can be computed from $N$ waveforms and their echoes is obtained by transmitting the singular functions corresponding to the $N$ largest singular values of two kernels derived from the target density. The singular functions are valid wavelets that obey an additional orthogonality constraint in the frequency domain. Using this result, the paper briefly discusses a solution to the problem of choosing a set of $N$ waveforms to reconstruct with high accuracy an arbitrary unknown target range-Doppler density function.

1 Introduction

This paper discusses the problem of reconstructing a target range-Doppler density function from a finite set of recorded echoes. The transmitted signals are assumed to be broadband. For simplicity of presentation, it is also assumed that perfect measurements of the echoes are available. The theory presented here will hold true if the observed echoes are corrupted by white noise. A simple modification can be used if the corrupting noise is colored and its covariance is known.

The above problem has not been widely treated in the literature. It was discussed earlier in [1] and [2]. Recent approaches to this problem include a tomographic based approach [3], [4], [5], a spectral estimation approach [6] and a group theoretical approach [7].

The main contribution of the paper is to show that the most accurate reconstruction of a range-Doppler target density that can be obtained using only $N$ waveforms and their echoes results from transmitting the singular functions corresponding to the $N$ largest singular values of two kernels derived from the target density. These singular functions are valid wavelets that obey an additional orthogonality constraint in the frequency domain. The paper then uses this result to briefly discuss a solution to the problem of choosing a set of $N$ waveforms to reconstruct with high accuracy an arbitrary unknown target range-Doppler density function.

Note that in practice the important question is how to construct the most accurate approximation to the range-Doppler target density by illuminating the target for a maximum of $T$ seconds. This problem is related to the one that is addressed here since the $N$ waveforms will have a total duration equal to the sum of their individual durations plus the duration of all silence intervals separating the waveforms. The length of these silence intervals is determined from a coarse a priori knowledge about the support of $D(x, y)$. If the radar selects the transmitted waveforms from a fixed library based on the approach presented here, the individual durations would be known a priori. Since the theory developed in this paper also identifies the relative importance of each transmitted waveform in the reconstruction, it is possible to use $T$ to select an appropriate subset of $N$ individual waveforms for optimal imaging of the target in less than $T$ seconds.

2 Problem Formulation

Consider first a point reflector at a distance $r(t_0)$ from a monostatic radar at time $t_0$. Let $s(t)$ be the waveform transmitted by the radar. We will assume here that the echo received by the radar at time $t$ is given by

$$A s(t - \tau(t))$$

where $A$ is a constant determined by the reflection properties of the point target and the propagation characteristics of the medium and $\tau(t)$ is the total delay incurred by the part of the waveform that arrives at the radar at time $t$. Note that

$$\tau(t) = \frac{2}{c} \left( t - \frac{1}{2} \tau(t) \right)$$

where $c$ is the velocity of propagation of the electromagnetic waves in the medium.

It is convenient to express $\tau(t)$ as a power series over the signal duration. In particular, if the change in the velocity of the reflector over the illumination time (duration of $s(t)$) is negligible compared to $c$, then we find that [8]

$$\tau(t) \approx x + \frac{2v(x/2)}{c + v(x/2)}(t - x)$$

where $x$ is an arbitrary reference time instant and $v(x/2)$ is the velocity of the point reflector at time $x/2$ along the line of sight. Note that it follows from (2) that the range of the target at time $t = x/2$ is $cx/2$.  

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Hence, the received waveform at time \( t \) will have the form

\[
\epsilon(t) = \int_0^\infty dy \int_{-\infty}^\infty dz \, D(x,y) \nu(y(t-x)).
\]  

(4)

In the above equation \( D(x,y) \) is the reflectivity of a point target located at range \( cz/2 \) and moving with velocity \( (1-y)/c(1+y) \) at time \( z/2 \). Since a negative range is meaningless, \( D(x,y) = 0 \) for all \( x < 0 \). Our goal in the next section will be to reconstruct the best approximation to \( D(x,y) \) by recording the echoes due to a fixed number \( N \) of transmitted waveforms.

3 Approximation of a known \( D(x,y) \) using a set of waveforms and echoes

First observe that (4) has the same form as an inverse wavelet transform. Specifically, if \( \psi(t) \) is a valid wavelet function then any square integrable signal \( f(t) \) can be written as

\[
f(t) = \frac{1}{C_\psi} \int_0^\infty dy \int_{-\infty}^\infty dz \, F(x,y) \sqrt{\psi(y(t-x))}
\]

(5)

where \( C_\psi \) is a constant that depends on \( \psi(t) \) and \( F(x,y) \) is the continuous wavelet transform of \( f(t) \) with respect to \( \psi(t) \). This observation seems to suggest then that one can reconstruct \( D(x,y) \) by transmitting a single valid wavelet function \( \psi(t) \) and computing the wavelet transform of the echo. Unfortunately, as observed in [7] and [9], following such an approach simply yields the projection of \( D(x,y) \) onto the range of the wavelet transform with respect to \( \psi(t) \).

Clearly the problem that we are addressing involves reconstructing a 2-D function from 1-D observations. Thus, in general, reconstructing an arbitrary \( D(x,y) \) will require transmitting and recording an infinite number of waveforms and echoes. Suppose on the other hand that we are restricted to using \( N \) or less waveforms. How can we optimize the choice of these waveforms to get the most accurate approximation of \( D(x,y) \) in the norm \( \|D(x,y)\|^2 = \int_0^\infty dy \int_{-\infty}^\infty dz |D(x,y)|^2 \)? We will answer this question by first assuming that \( D(x,y) \) is actually known. The answer that we obtain will then guide our choice of transmitted waveforms for the actual case where \( D(x,y) \) is unknown.

By taking the Fourier transform of (4) we obtain

\[
E(\omega) = \int_0^\infty \Delta(\omega,y) S(\frac{\omega}{y}) \frac{dy}{y}.
\]  

(6)

In the above equation \( \Delta(\omega,y) \) is the Fourier transform of \( D(x,y) \) with respect to \( x \), i.e.

\[
\Delta(\omega,y) = \int_{-\infty}^\infty D(x,y) e^{-j\omega x} dz.
\]  

(7)

Define \( E_\pm(\omega) = E(\omega) \) for \( \omega \geq 0 \) and \( E_\pm(\omega) = 0 \) otherwise. Similarly, let \( E_-(\omega) = E(\omega) \) for \( \omega \leq 0 \) and \( E_-(\omega) = 0 \) otherwise. Now let us change the variable of integration by defining \( u = \omega/y \). With this change of variable of integration (6) yields the following two integrals

\[
E_+(\omega) = \int_0^\infty T_+(\omega,u) S(u) \frac{du}{u}
\]  

(8)

and

\[
E_-(\omega) = \int_0^\infty T_-(\omega,u) S(u) \frac{du}{|u|}.
\]  

(9)

In the above equations \( T_+(\omega,u) = T(\omega,u) \) for \( \omega \geq 0 \) and \( T_-(\omega,u) = 0 \) otherwise, \( T_-(\omega,u) = T(\omega,u) \) for \( \omega \leq 0 \) and \( T_+(\omega,u) = 0 \) otherwise and \( T(\omega,u) = \Delta(\omega,u) \). Note that \( T_+(\cdot,\cdot) \) are kernels of two maps that take \( L_2^2(\mathbb{R}_+, du/|u|) \) into \( L_2^2(\mathbb{R}_+, dw) \). If we assume that \( D(\omega) \) has finite energy, i.e. if

\[
\int_{-\infty}^\infty dz \int_0^\infty dy |D(x,y)|^2 < \infty
\]  

(10)

then it may be shown that the operators corresponding to \( T_\pm(\cdot,\cdot) \) are compact operators. Hence these operators admit a singular value decomposition. In particular, it follows that the best approximations to either \( T_+(\omega,u) \) or \( T_-(\omega,u) \) using \( M \) functions \( S_\pm(u) \), \( n = 1, M \) and their corresponding echoes (images under \( T_\pm(\cdot,\cdot) \)) \( E_\pm(\omega) \) is obtained by choosing the functions \( S_\pm(u) \), \( n = 1, M \) to be the singular functions of \( T_+(\omega,u) \) or \( T_-(\omega,u) \) corresponding to their \( M \) largest singular values. Note that since the singular functions \( S_\pm(u) \) of \( T_\pm(\cdot,\cdot) \) belong to \( L_2^2(\mathbb{R}_+, du/|u|) \) they are the Fourier transforms of valid wavelet functions. Specifically, we have

\[
\int_0^\infty |S_\pm(u)|^2 du < \infty
\]  

(11)

which is the admissibility condition for wavelet functions.
largest singular values. The precise choice is made by computing the reduction that will occur in the norm of the error by including a particular singular function of $T_+(\omega, u)$ versus one of $T_-(\omega, u)$. The singular functions are considered in the order in which their corresponding singular values appear when arranged in order of decreasing magnitude.

4 Reconstruction of an unknown $D(x, y)$

The results of the last section provide a general guideline for choosing $N$ waveforms for optimal reconstruction of a range-Doppler distribution $D(x, y)$. Specifically, the waveforms should be close approximations to the singular functions of $T_+(\omega, u)$ and $T_-(\omega, u)$ corresponding to their $N$ largest singular values. The problem of course is that $D(x, y)$ is unknown.

It turns out that it is possible to find a set of functions that would act as approximate singular values for all kernels $T_+(\omega, u)$ and $T_-(\omega, u)$ that correspond to finite support range-Doppler densities $D(x, y)$. In particular, if the Fourier transforms $S_\pm(u)$ are chosen such that each $S_\pm(ln(u))$ is equal to a particular translate and dilate of the same discrete wavelet function with a large number of vanishing moments then they will provide a good approximation to the singular functions of all kernels $T_+(\omega, u)$ and $T_-(\omega, u)$ that correspond to finite support range-Doppler densities $D(x, y)$ [10].

We still need to impose an ordering on these approximations to the singular functions of the kernels $T_+(\omega, u)$ and $T_-(\omega, u)$ that we expect to encounter. Specifically, we are really interested in approximating the singular functions of $T_+(\omega, u)$ and $T_-(\omega, u)$ corresponding to their largest singular values. This can again be done using the fact that $D(x, y)$ has a finite support in the $(x, y)$ plane. The finite support constraint implies that $\Delta(\omega, y)$ is a smooth function in $\omega$ (its Fourier transform is a low pass function) and has a finite support in the $y$ variable. This fact enables us to predict the asymptotic behavior of the wavelet coefficients in a 2-D discrete wavelet decomposition of either $T_+(\omega, u)$ or $T_-(\omega, u)$ [10].

More generally, if we can collect data corresponding to several representative $D(x, y)$ profiles we may use the following two step adaptive range-Doppler imaging scheme. In the first step we classify the available $D(x, y)$ densities into several classes using a clustering algorithm based on a norm criterion, e.g. the $L^2$-norm. Next, for each class we compute a set of $N$ waveforms that act as approximations to the singular functions corresponding to the largest singular values of the kernels $T_+(\omega, u)$ and $T_-(\omega, u)$ in this class. We also construct a set of $N$ fixed waveforms to be used in a pre-imaging classification step. We will refer to these waveforms as the classification waveforms. This first step is done off-line.

The second step is performed on line during actual imaging of a target. The radar first transmits the classification waveforms. A vector quantization routine then uses the approximate $D(x, y)$ reconstructed using these waveforms to determine the class of range-Doppler densities to which the observed $D(x, y)$ belongs. The radar finally transmits the appropriate set of waveforms corresponding to the identified class to obtain a higher resolution image of $D(x, y)$. The details of this procedure are given in [10].

5 A Simulation Example

Let us illustrate the above technique with a simple example. Assume that it is desired to image the distribution $D(x, y)$ shown in Fig. 1. This distribution consists of two 2-D Gaussian functions. The optimal reconstructions that we can obtain by sending one or two properly chosen wavelets are shown in Figs. 2 and 3. Note that Fig. 3 is essentially $D(x, y)$. This is the case because $D(x, y)$ is actually a rank 2 kernel. Hence, it can be reconstructed exactly using two properly chosen waveforms.

References

Figure 1: Actual $D(x,y)$

Figure 2: Reconstructed $D(x,y)$ using 1 wavelet

Figure 3: Reconstructed $D(x,y)$ using 2 wavelets