Abstract—A novel autocorrelation estimator is developed using Slepian sequences as multiple windows, which has more degrees of freedom than any single-window estimate, including the sample average, with the same frequency domain resolution. Because the Slepian sequences are orthogonal, confidence intervals can be estimated by jacknifing as well as by standard $\chi^2$ methods. The proposed multiple window estimator is applied to batch AR parameter estimation and recursive least squares equalization. Both applications show significant improvement, especially for small data lengths, while only linearly (in the number of windows) increasing computational complexity. Generalizations to higher-order correlation estimators are delineated.

1 Introduction

Recently many extensions and applications of Thomson's multiple window (MW) spectrum estimator [3] have appeared [4, 8, 11]. All these ideas begin with the Cramér representation, which relates a random process to its underlying power spectrum, and proceeds to a multiple window spectrum estimator, which relates finite random data with its underlying estimated power spectrum.

In Section 2 of this paper, a time domain approach is taken, and in Section 3 it is shown that the resulting sample autocorrelation is well estimated directly from the time domain data using multiple windows. This estimate is expected to have more degrees of freedom (and hence lower variance) than any single-window estimate, including the traditional sample average. This increase in stability does not cause any loss in resolution. Cross correlations can also be estimated with multiple windows. Use of multiple window correlation estimates may improve the performance of algorithms which are based on sample autocorrelations, while at most linearly (in the number of windows) increasing the computational complexity. The multiple window estimate can be recursively updated for use in adaptive algorithms. We provide modified Recursive Least Squares (RLS) and Levinson-Durbin algorithms which employ the new estimator in Section 4, and show simulation results in Section 5. Noticable improvements in AR parameter estimation and RLS convergence are verified by simulations.

2 MW Correlation Estimator

In [3], Thomson showed that the Cramér representation of a stationary random process $\{u(n)\}$,

$$u(n) = \int_{-1/2}^{1/2} e^{j2\pi fn} dZ(f),$$

(1)

can be used to derive an approximate expression for the power spectral density $S_{uw}(f) = \mathbb{E}\{|dZ(f)|^2\}$ given finite data $u(0), \ldots, u(N - 1)$. This approximate expression is derived from the fact the Slepian functions are eigenfunctions of the Dirichlet kernel, and thus naturally arise in the solution of the integral equation (1). We denote the $k$th discrete prolate spheroidal, or Slepian, sequence by $v_k(n)$, with implicit dependence on $N$, the data length, and $W$, the half-bandwidth of $u(n)$. The Slepian sequences, with their corresponding eigenvalues $\lambda_k$, satisfy the eigenvector relation

$$Dv_k = \lambda_k v_k,$$

where $v_k \triangleq [v_k(0), \ldots, v_k(N - 1)]^T$, and $D$ is the matrix with entries

$$d_{mn} = \frac{\sin\{2\pi W(n - m)\}}{\pi(n - m)}.$$

The sequences $v_k(n)$ are ordered by their eigenvalues, so that $1 > \lambda_0 > \lambda_1 > \cdots > \lambda_{N-1} > 0$. Let $w_k(n) = u(n)v_k(n)$ be the data $u(n)$ windowed by the $k$th Slepian sequence, and $U_k$ be the Fourier transform...
Thomson proposed as an estimate of $S_{uw}(f)$ the following weighted eigenspectrum average:

$$S_{uw}^{\text{mw}}(f) = \frac{1}{K} \sum_{k=0}^{K-1} \frac{1}{\lambda_k} |U_{uw}(f)|^2,$$  

(2)

where the number of windows, $K$, is typically $2NW$. As noted in [3, 6], $S_{uw}^{\text{mw}}(f)$ is consistent, with variance $O(N^{-1})$, and approximately $2K$ degrees of freedom. The estimator in (2) also has the same frequency resolution as a single window estimate, which has 2 degrees of freedom.

The multiple window spectrum estimate (2) can be jacknifed to provide confidence intervals [7]. Jacknifed confidence intervals can be more accurate than standard intervals computed from $\chi^2$ distributions, since the latter method is independent of the data, and assumes Gaussianity with no outliers [7].

Motivated by the desirable statistical properties of the multiple window spectrum estimator, we propose as an estimate of the autocorrelation $R_{uu}(\tau)$ the inverse Fourier transform of $S_{uw}^{\text{mw}}(f)$ in (2); i.e.,

$$\hat{R}_{uu}^{\text{mw}}(\tau) = \mathcal{F}^{-1}[S_{uw}^{\text{mw}}(f)].$$

This can be expressed in the time domain as

$$\hat{R}_{uu}^{\text{mw}}(\tau) = \frac{1}{K} \sum_{k=0}^{K-1} \left[ \frac{1}{\lambda_k} \sum_{t=0}^{N-1} u_{wk}(t)u_{wk}(t+\tau) \right].$$

(3)

Similarly, the cross correlation between two signals $\{x(n)\}$ and $\{y(n)\}$ can be estimated by

$$\hat{R}_{xy}^{\text{mw}}(\tau) = \frac{1}{K} \sum_{k=0}^{K-1} \left[ \frac{1}{\lambda_k} \sum_{t=0}^{N-1} x_{wk}(t)y_{wk}(t+\tau) \right],$$

(4)

where $x_{wk}$ and $y_{wk}$ are the data windowed by the $k$th Slepian sequence. The estimator $\hat{R}_{xy}^{\text{mw}}(\tau)$ is a weighted average of $K$ sample autocorrelations with the data for each windowed by the $k$th Slepian sequence.

3 Variance of $\hat{R}_{uu}^{\text{mw}}(\tau)$

With the assumption that $\{u(n)\}$ is stationary up to order four, the variance of the traditional sample average estimator

$$\hat{R}_{uu}(\tau) = \frac{1}{N} \sum_{t=0}^{N-1} u(t)u(t+\tau)$$

(6)

can be expressed as (see, e.g., [2, p. 325])

$$\text{var}\{\hat{R}_{uu}(\tau)\} = \frac{1}{N^2} \sum_{t=0}^{N-1} \sum_{s=0}^{N-1} [R_{uu}^2(s-t) + R_{uu}(s-t+\tau)R_{uu}(s-t-\tau) + \kappa_4(s-t, \tau, 0)],$$

(7)

where $\kappa_4(s-t, \tau, 0)$ is the fourth order cumulant of $\{u(n)\}$ (if $\{u(n)\}$ is Gaussian, this term is zero). Similarly, the variance of (6) when the data is windowed by a single window $w(t)$ is

$$\text{var}\{\hat{R}_{uu}^{\text{mw}}(\tau)\} = \frac{1}{N^2} \sum_{t=0}^{N-1} \sum_{s=0}^{N-1} [R_{uu}^2(s-t) + R_{uu}(s-t+\tau)R_{uu}(s-t-\tau) + \kappa_4(s-t, \tau, 0)] \times [w(t)w(t+\tau)w(t+\tau)].$$

(8)

Now, denoting the $k$th Slepian-windowed correlation estimate as

$$\hat{R}_{k}(\tau) = \sum_{t=0}^{N-1} u(t)u(t+\tau)v_k(t)v_k(t+\tau),$$

(9)

and including only those windows for which $\lambda_k \approx 1$, the multiple window correlation estimator can be rewritten as

$$\hat{R}_{uu}^{\text{mw}}(\tau) = \frac{1}{K} \sum_{k=0}^{K-1} \hat{R}_{k}(\tau).$$

Now we assume that the data is Gaussian to find that

$$\text{cov}\{\hat{R}_{k}(\tau), \hat{R}_{j}(\tau)\} = \frac{1}{N^2} \sum_{t=0}^{N-1} \sum_{s=0}^{N-1} [R_{uu}^2(s-t) + R_{uu}(s-t+\tau)R_{uu}(s-t-\tau)] \times [v_k(t)v_k(t+\tau)v_j(t+\tau)].$$

The variance of $\hat{R}_{uu}^{\text{mw}}(\tau)$ is therefore given by

$$\text{var}\{\hat{R}_{uu}^{\text{mw}}(\tau)\} = \frac{1}{(KN)^2} \sum_{k=0}^{K-1} \sum_{j=0}^{K-1} \sum_{t=0}^{N-1} \sum_{s=0}^{N-1} [R_{uu}^2(s-t) + R_{uu}(s-t+\tau)R_{uu}(s-t-\tau)] \times [v_k(t)v_k(t+\tau)v_j(t+\tau)].$$

(10)

If no windowing is performed (i.e., $K = 1$ and a rectangular window is used), (10) reduces to (6), as expected. Due to the orthogonality of the windows, the individual $\hat{R}_k(\tau)$ estimates are approximately uncorrelated. This property produces the same type of variance reduction as subdividing the data and averaging, but since each $\hat{R}_k(\tau)$ uses all of the data, the bias is not increased.
4 Higher order correlations

Thomson's spectrum estimator (2) can be generalized to polyspectra (see [4]). The bispectrum estimator, for example, is

\[
\hat{S}_{3u}(f_1, f_2) = \frac{1}{\gamma} \sum_{j,k,l=0}^{K-1} U_{w_j}(f_1)U_{w_k}(f_2)U_{w_l}(f_3)P(j, k, l),
\]

where \(P(j, k, l) = \sum_{n=0}^{N-1} v(j(n))v(k(n))v(l(n)), \gamma = \sum_{j,k,l=0}^{K-1} P^2(j, k, l), \text{and } (f_1 + f_2 + f_3) \mod \frac{1}{2} = 0.

By taking the two dimensional inverse Fourier transform of the bispectrum estimator, the third order moment can be estimated for \(0 \leq \tau_2 \leq \tau_1, \gamma\)

\[
\hat{n}_{3u}^{MW}(\tau_1, \tau_2) = \frac{1}{\gamma} \sum_{j,k,l=0}^{K-1} \left[ \sum_{t=0}^{N-\tau_1-1} u_{w_j}(t)u_{w_k}(t+\tau_1)u_{w_l}(t+\tau_2) \right] \times P(j, k, l).
\]

The improved performance for short data lengths is especially important for higher-order estimators, due to their intrinsically higher variances. Unfortunately, the computational complexity is not proportional to \(K\) but increases as a power of \(K - 1\).

5 Applications

The multiple window \(\hat{R}_{3u}^{MW}(\tau)\) can be used to improve algorithms relying on correlation estimates. We have developed modified AR parameter estimation and RLS algorithms which use this new estimate. For AR parameter estimation, the new autocorrelation estimate is used to solve the Yule-Walker equations for the AR coefficients with the Levinson-Durbin algorithm. \(K\) applications of the Levinson-Durbin algorithm are performed, each using a different Slepian window for the data. The resulting AR parameter estimates are then averaged to form the multiple window AR estimate. Due to the linearity of the algorithm, this is equivalent to solving the Levinson-Durbin algorithm with the MW correlation estimator.

Recursive updates are needed for adaptive algorithms such as the RLS. The multiple window auto- and cross-correlation matrices can be recursively updated as:

\[
\hat{R}_{3u}^{MW}(n) = \hat{R}_{3u}^{MW}(n-1) + \sum_{k=0}^{K-1} \frac{1}{\lambda_k} u_{w_k}(n)u_{w_k}^H(n),
\]

where \(u_{w_k}(n)\) is a vector of data \(u(n) = [u(n), \ldots, u(n-M+1)]^T\) windowed by the \(k\)th Slepian sequence, and \(\hat{R}_{3u}^{MW}(n)\) is the estimated MW correlation matrix. A similar update is used for cross correlation updates. \(K\) applications of the matrix inversion lemma are used to produce a recursive cross correlation matrix necessary for the RLS algorithm.

6 Simulations

Two simulations are shown demonstrating the improvement of the MW estimator over the conventional sample average estimator. In test case one, the output of an AR(14) model is used to form autocorrelation estimates. Two correlation estimates are computed, one with a Hamming window, the other using five Slepian sequences. In each trial, the Euclidean distances from the true AR parameters was summed for both estimators. For each data length, 50 Monte Carlo runs were performed. Means and variances were calculated and are shown in Figure 1 as a function of data length.

The improved performance shown in Figure 1 is dramatic. In the graph, the mean distance from the true parameters is plotted along with a one standard-deviation bound. The upper set of curves is from the Hamming window, the lower is from the multiple windows. Using five windows, a data length of 20 can be used to achieve the mean error of one Hamming window with a data length of 140. The performance compares even more favorably with the rectangular window, which corresponds to the conventional estimator in (1). The use of such a short data window is crucial for the analysis of non-stationary signals, such as speech.

RLS improvement is equally impressive. Here the RLS algorithm is equalizing an FIR filter given the input and the output sequences. A twelfth order FIR filter is used for equalization. The mean Euclidean distance from the optimum tap weights is plotted over 40 trials against iteration number. The upper curve is from the conventional RLS algorithm, the lower is from our multiple window algorithm. RLS is expected to converge in approximately \(2M\) iterations, where \(M\) is the FIR length. The multiple window RLS converges in about two-thirds the number of iterations, using three Slepian windows.

7 Conclusions

Simulations indicate that the MW correlation estimator provides a lower variance estimator without increasing the bias. Although it is reasonable to assume that if \(\hat{S}_{3u}^{MW}(f)\) is a "good" estimate of \(S_{3u}(f)\), then the inverse Fourier transform of \(\hat{S}_{3u}^{MW}(f)\) is a good estimate of the inverse transform of \(S_{3u}(f)\), such an argument is still ad-hoc. The MW estimator \(\hat{R}_{3u}^{MW}(\tau)\) should be the result of an optimality criterion which
utilizes the unique properties of the Slepian functions. Derivation of such an optimality measure, along with jacknifed variance expressions, are interesting future research topics.

References


