Convergence Analysis for the Extended-Kalman-Filter Based Algorithm of Target Tracking and Identification

Dan S. Cheng and Allen R. Stubberud
Center for High-Speed Image/Signal Processing (CHIP)
Department of Electrical and Computer Engineering
University of California, Irvine
Irvine, CA 92717

Abstract
The joint problem of target tracking and identification has been discussed in [1]. The tracking/identification algorithm using an extended-Kalman-filter-based associative memory (EKFAM) has been demonstrated through several examples. In this paper, we discuss the convergence properties of the algorithm. Under appropriate conditions, a contraction operator can be developed using Banach space concepts that guarantee convergence of the algorithm.

Some Conditions for the Convergence Algorithm
The algorithm of the EKFAM offers a rapid technique for target tracking and identification [1]. In this paper, we show that under certain conditions, the algorithm is guaranteed to give convergent results. The appropriate analysis, based on Banach space methods, is provided for the norm of the error covariance of the extended Kalman filter. The condition needed to guarantee that the norm of the sequence of error covariance operators is non-increasing and less than one is developed. More specifically, a contraction operator is developed for the error covariance. Some of the analysis necessary to show that the EKFAM estimates converge is given below:

Definition 1
The norm of a bounded linear operator $A : X \rightarrow Y$, where $X, Y$ are normed linear vector spaces, is defined $\forall z \in X$ by:

$$
\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.
$$

Following Definition 1, we can define an $N \times M$ matrix $B$ norm as

$$
\|B\| = \sqrt{\|B^TB\|}.
$$

We let $A = B^TB$, then

$$
\|A\| = \max_{\|x\|=1} \|Ax\| = \max_{\|x\|=1} \|\lambda x\| = \max |\lambda|,
$$

where $\lambda$ is an eigenvalue of $A$.

Theorem 1
If $\{A_m\}$ is a sequence of linear operators on $X$ (i.e., maps $X$ into itself), $X \subset R^m$, such that $\|A_m\| < 1$, then there exists a contraction operator $A$ such that

$$
\|A_m - A\| \rightarrow 0, \text{ as } m \rightarrow \infty.
$$

Theorem 2 (Neumann Series)
Suppose $A$ is a contraction operator, that is, $\|A\| < 1$, and $A : X \rightarrow X$ is a mapping from Banach space $X$ into itself, and let $I : X \rightarrow X$ be an identity operator. Then: $I - A$ has a bounded inverse operator on $X$ which is given by the Neumann Series

$$
(I - A)^{-1} = \sum_{k=0}^{\infty} A^k
$$

and which satisfies

$$
\| (I - A)^{-1} \| \leq \frac{1}{1 - \|A\|}
$$

the iterated operators $A^n$ are defined by $A^0 = I$ and $A^n = A \cdot A^{n-1}$ for $n \in N$, the set of natural numbers.
numbers. Now we can apply definition 1 and theorems 1 and 2 to our problem. The system to be identified in [1] was defined by:

\[ X(k+1) = A(k)X(k) + w_T(k), \]

where

\[ A(k) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \Delta_k \\ 0 & 0 & 1 \end{pmatrix}, \]

where \( \Delta_k \) was chosen to be small and decreasing, and the measurement of the target feature vectors was given by:

\[ F(k+1) = h(X(k+1)) + v(k) \]

Let \( H(k+1) \) be the Jacobian matrix of \( h(X(k+1)) \). \( H(k) \) and \( v(k) \) are assumed to be zero mean white noises with covariance \( Q_k \) and \( R_k \), respectively. Thus, from the extended Kalman filter literature, we have the following equations for the calculation of error covariances:

\[ P(k+1|k) = A(k) \cdot P(k|k) + P(k|k) \cdot A^T(k) + Q_k \]

\[ K(k+1) = P(k+1|k) \cdot H(k+1) \]

\[ = [I - k(k+1) \cdot H(k+1)] \cdot P(k+1|k) \cdot [I - k(k+1) \cdot H(k+1)]^T + K(k+1) \cdot R \cdot K^T(k+1) \]

where \( P(k+1|k) \) is the prediction error covariance, \( K(k+1) \) is the Kalman gain, and \( P(k+1|k) \) is the error covariance. The sequence of norms of the error covariance are shown, under certain conditions, to be a sequence of contraction operators so that the convergent algorithm exists. Now, we use the analysis to show that the algorithm developed in [1] is convergent under the conditions as described below:

Suppose \( Q_0, P_0 \) are both positive definite diagonal square matrices. We assume the largest eigenvalues for \( Q_0 \) and \( P_0 \) are \( \lambda_{q0} \) and \( \lambda_{p0} \), respectively. We know from the above definition that \( \|Q_0\| = \lambda_{q0}, \|P_0\| = \lambda_{p0} \). Also \( \|A_k\| = 1 \). Since \( H(k+1) \cdot P(k+1|k) \cdot H^T(k+1) \) is a nonnegative definite \( n \times n \) matrix, where \( n \) is the number of the elements of the target feature vector. In our case, \( H(k+1) \) is defined as \( n \times 3 \) matrix and we found the norm \( \|H\| \leq 1 \). Notice that \( H(k+1) \cdot P(k+1|k) \cdot H^T(k+1) \) is a non-negative singular matrix, thus its smallest eigenvalue is zero. Let \( R \) be an \( n \times n \) positive definite diagonal matrix, then \( H(k+1) \cdot P(k+1|k) \cdot H^T(k+1) + R \) is a positive definite matrix. Denote \( R_{min} \) and \( R_{max} \) as the smallest and largest eigenvalues for the \( R \) matrix. For

\[ R_{max} \geq R_{min} > 1 \]

we have:

\[ P(1|0) = A_0 \cdot P(0|0) + P(0|0) \cdot A_0^T + Q_0 \]

\[ K(1) = P(1|0) \cdot H^T(1) \cdot [H \cdot P \cdot H^T + R]^{-1} \]

\[ = [I - K(1) \cdot H(1)] \cdot P(1|0) \cdot [I - K(1) \cdot H(1)]^T + K(1) \cdot R \cdot K^T(1) \]

where \( [H \cdot P \cdot H^T + R]^{-1} \) is of the form \( \beta_k \), and \( \lambda_{p1} = \|P(k+1|k)\|, \lambda_{p2} = \|P(k|k)\|, \) and \( 0 < \gamma_k < \beta_k \cdot \alpha \), then

\[ \beta_0 \leq 2\lambda_{p0} + \lambda_{q0} \]

\[ \lambda_{p1} \leq (1 - \gamma_1)^2 \cdot \beta_0 + \beta_0^2 \cdot \alpha^2 \cdot R_{Max} \]

\[ \beta_1 \leq 2\lambda_{p1} + \lambda_{q1} \leq 2[(1 - \gamma_1)^2 \cdot \beta_0 + \beta_0^2 \cdot \alpha^2 \cdot R_{Max}] + \lambda_{q1} \]

and

\[ \|K(2)\| \leq \alpha \cdot \beta_1 \]

\[ \lambda_{p2} \leq (1 - \gamma_2)^2 \cdot \beta_1 + \beta_1^2 \cdot \alpha^2 \cdot R_{Max} \]

\[ \vdots \]

\[ \lambda_{p\delta} \leq (1 - \gamma_{\delta})^2 \cdot \beta_{\delta-1} + \beta_{\delta-1}^2 \cdot \alpha^2 \cdot R_{Max} \]

We want to construct the contraction operators such that \( \lambda_{p\delta} \leq 1, \forall \delta \), and \( \lambda_{p\delta} \) is a non-increasing sequence, then according to Theorem 1, we know that the extended Kalman filter algorithm will converge. First, we use a minimum-maximum technique to estimate the \( \lambda_{p1} \). Since \( (1 - \gamma_1)^2 \cdot \beta_0 + \beta_0^2 \cdot \alpha^2 \cdot R_{Max} \) is the upper bound for \( \lambda_{p1} \), then its minimum is 2 \( \beta_0^2 \cdot \alpha^2 \cdot R_{Max} \)

\[ \Rightarrow (1 - \gamma_1)^2 \leq \beta_0 \cdot \alpha^2 \cdot R_{Max} \]

For

\[ 2 \cdot \beta_0^2 \cdot \alpha^2 \cdot R_{Max} < 1, \]
\[
\Rightarrow \beta_0 < \frac{1 + R_{\text{Max}}}{\sqrt{2}R_{\text{Max}}}
\]

then

\[
(1 - \gamma_1)^2 = \beta_0 \cdot \alpha^2 \cdot R_{\text{Max}} < \frac{\sqrt{2}R_{\text{Max}}}{2R_{\text{Min}}}
\]

Thus

\[
\gamma_1 > 1 - \frac{\sqrt{2}R_{\text{Max}}}{\sqrt{2}R_{\text{Min}}}
\]

and

\[
R_{\text{Max}} < 2 \cdot R_{\text{Min}}^2
\]

Since \( \gamma_1 < \alpha \cdot \beta_0 \)

\[
\Rightarrow \beta_0 > (1 - \frac{\sqrt{2}R_{\text{Max}}}{\sqrt{2}R_{\text{Min}}})(1 + R_{\text{Max}})
\]

and

\[
\beta_0 \leq 2l_{p_0} + \lambda_{q_0}
\]

then the condition number one for \( P(1|1) \) to be a contraction operator is

\[
(1 - \frac{\sqrt{2}R_{\text{Max}}}{\sqrt{2}R_{\text{Min}}})(1 + R_{\text{Max}}) < 2l_{p_0} + \lambda_{q_0}
\]

Similarly, by taking the minimum upper bounds that we could find

\[
\lambda_{p_2} \leq 2\beta_0^2 \cdot \alpha^2 \cdot R_{\text{Max}}
\]

\[
\vdots
\]

\[
\lambda_{p_n} \leq 2\beta_0^2 \cdot \alpha^2 \cdot R_{\text{Max}}
\]

then,

\[
\lambda_{p_2} \leq 2[2 \cdot (2l_{p_0} + \lambda_{q_0})^2 \cdot \frac{R_{\text{Max}}}{R_{\text{Min}}} + \lambda_{q_1})^2 \cdot \frac{R_{\text{Max}}}{R_{\text{Min}}}
\]

Since

\[
\lambda_{p_1} \leq 2 \cdot \beta_0^2 \cdot \alpha^2 \cdot R_{\text{Max}}
\]

Choose then

\[
\frac{R_{\text{Max}}}{R_{\text{Min}}} < 1
\]

(2)

Thus we know that the contraction operator is developed if the relationship for the upper bound satisfies the following:

\[
2 \cdot [2 \cdot (2l_{p_0} + \lambda_{q_0})^2 \cdot \lambda_{q_1}]^2 
\]

\[
\leq 2 \cdot (2l_{p_0} + \lambda_{q_0})^2 < 1
\]

(3)

The above inequality [(1), (2), (3)] relationships will also guarantee the norm of the sequence \( P(k+1|k) \) of operators is nonincreasing. Namely, the algorithm for the target identification and tracking is convergent.

References


