DIGITAL T/2 NYQUIST FILTERING USING RECURSIVE ALL-PASS TWO-STAGE RESAMPLING FILTERS FOR A WIDE RANGE OF SELECTABLE SIGNALLING RATES

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1. ABSTRACT
Digital filters process sampled data signals to perform the many tasks implemented in a conventional analog receiver. The sample rate of the processing is selected to match the bandwidth of the signal and an analog anti-aliasing filter precedes the sampler to assure compliance with the Nyquist criterion. When the system operates over a large span of data rates, hence bandwidths, a combination of variable analog bandwidth, variable sample rate, and digital resampling filter is required to obtain any desired output sample rate. We describe a design and implementation of a cost effective polyphase recursive resampling filter which affords very high resolution adjustable sample rate as required in a high performance MODEM.

2. INTRODUCTION
A MODEM, the receiver used for demodulating digital data, is traditionally implemented with analog circuitry. With the advent of inexpensive Application Specific Integrated Circuits (ASIC) and DSP chip sets, many of the functions of a traditional MODEM are being implemented by DSP hardware and software. The usual advantage cited for a DSP implementation is the flexibility afforded by the option to vary the characteristics of a filter by simply downloading new filter coefficients.

A unique advantage of a DSP solution is the opportunity to change the sample rate as part of the signal processing. This is particularly important if the MODEM is designed to operate over a wide range of data rates. A straightforward but expensive solution to this problem is to implement a variable anti-aliasing filter and sample rate to match each input bandwidth. A more sophisticated approach holds the analog filter and sample rate fixed and uses multi-rate signal processing algorithms to accommodate the different bandwidths and sample rates. From one viewpoint this is an elegant solution since it is easy to verify that any rational ratio of input to output sample rates can be obtained by resampling filters. From another viewpoint, the computational load and control complexity required to support an arbitrary ratio may be formidable and the amount of memory required to hold the necessary coefficient sets may be unacceptably large.

A third option is a hybrid combination of the two previous choices. This option permits a programmable change in sample rate over a restricted range while supporting digital resampling filters which efficiently change sample rates for selected, easy to implement, input-output ratios. All three options are presented in figure 1 and this paper addresses the last option.

3. BACKGROUND
A communication system communicates through a noisy bandlimited channel. To ameliorate the effects of noise (and out of band interference), the receiver must contain a filter matched to the transmitted waveshape. Usually the transmitter and receiver contain the same filter; one at the transmitter generating the waveform in response to the input data, and one at the receiver performing the matched filtering operation. To minimize intersymbol interference, the signal processed by the detector must be a Nyquist waveform, one having equally spaced zero crossings matched to the channel
data rate of \( T \)-seconds. A common realization of a Nyquist pulse (or waveshape) has a power spectrum exhibiting a raised-cos taper with the point of odd symmetry in the taper located at frequency \( 0.5/T \) Hz.

To satisfy requirements of both matched filtering and Nyquist pulse shaping, the filters at the transmitter and receiver are designed to have a power spectrum with the square root of the desired cosine taper spectrum. This filter, called the root-raised-cosine filter, exhibits an amplitude gain at the \( 0.5/T \) Hz design frequency of \( \sqrt{0.5} \) or 0.707 which is commonly denoted the half power frequency. A Butterworth filter with its 3-dB bandwidth matched to the channel data rate is a good approximation to this filter. The minimum (Nyquist) sample rate required at the output of these filters is, of course, the channel data rate of \( T \) seconds per sample. To permit clock acquisition and to support channel equalization, data is often sampled at twice the minimum, or at \( T/2 \) seconds per sample. As a result, the 3-dB edge of the Butterworth filter is located at the quarter sample rate of the output sampling process. Such a filter is called a half band filter.

When we use the Bilinear Z-Transform to map the roots of a half-band Butterworth filter to the complex \( Z \)-plane we encounter an interesting property of the Bilinear Transform. The property is that the image of the unit circle (on which the normalized Butterworth roots of a half band filter are located) in the complex \( S \)-plane is the strip on the imaginary axis contained within the unit circle in the complex \( Z \)-plane. This leads to the peculiar result that the roots of all half-band Butterworth filters are on the imaginary axis, or equivalently, have zero real parts. This has an interesting impact on the implementation of the filter.

When a recursive filter is implemented as a product or sum of second order sections each section will have the transfer function indicated in (1). This transfer function requires 4 multiplications and 4 additions per output point for filters with zeros on the unit circle (ie., \( b_2 = 1.0 \)).

\[
F(Z) = b_2 \frac{Z^2 + b_1 Z + b_0}{Z^2 + a_1 Z + a_2} \tag{1}
\]

When the half-band Butterworth filter is implemented in this structure, the transfer function reverts to that of (2) which requires 3 multiplications and 3 additions per output point.

\[
F(Z) = b_0 \frac{Z^2 + b_2 Z + 1}{Z^2 + b_1 Z + a_2} \tag{2}
\]

where \( b_0 = \frac{1 + a_2}{2 + b_1} \).

We are including the \( b_1 \) coefficient in the work count to permit the option of implementing other filters with stopband zeros on the unity circle, such as the inverse Tchebyshev or Elliptic.

The lesson learned from this section is that the root-raised-cos filter is characterized by an interesting pole-zero placement and that computational saving can be had by taking advantage of that placement. A distinctive characteristic of this pole placement is that the denominator of the filter is a polynomial in \( Z \). This leads to a useful signal processing advantage which we will discuss in the next section.

4. THE LESSON OF POLE-ZERO PLACEMENT

We have earlier described [1,2] filter implementations using two-path recursive all-pass filters to form efficient half-band lowpass filters. The structure of this class of filters is shown in figure 2 and has the transfer function presented in (3).

\[
T(Z) = P_0(Z^2) + P_1(Z^2) Z^{-1} \tag{3}
\]

\[
P_0(Z^2) = \prod_{k=0}^{K} H_{i,k}(Z^2), \quad i=0,1
\]

The all-pass stages within two-path structure have the transfer function shown in (4) and can be implemented with single multiplier structures shown in figure 3.

\[
H(Z) = \frac{1 + \alpha Z^2}{Z^2 + \alpha} \tag{4}
\]
To demonstrate the organization of these filters we examine the case of a single stage in path-0 and only delay in path-1. The transfer function of this case is shown in (5). Note that the delay required in path-1 can be obtained from the delay in the all-pass stage of path-0 when implemented with the two delay structure of figure 3.

\[
H(Z) = \frac{1 + \alpha_0 Z^2}{Z^2 + \alpha_0} + \frac{1}{Z} = \frac{\alpha_0 Z^3 + Z^2 + \alpha_0}{Z(Z^2 + \alpha_0)} \quad (5)
\]

\[
\alpha_0 (Z-1) (Z^2 + (-1) + 1)
\]

\[
Z(Z^2 + \alpha_0)
\]

This simple transfer function is third order and for 0.3333.. < \alpha_0 < 1.0, the filter zeros are on the unit circle. This filter thus realizes three poles and three zeros with only a single multiplication. Not too bad!

In a similar fashion a filter with a single all-pass stage per path results in the fifth order transfer shown in (6). Here the delay of the path-1 stage can be obtained from the delay in the all-pass stage of path-0.

\[
H(Z) = \frac{1 + \alpha_0 Z^2}{Z^2 + \alpha_0} + \frac{1 + \alpha_1 Z^2}{Z^2 + \alpha_1}
\]

\[
\alpha_0 Z^5 + \alpha_1 Z^4 + (1 + \alpha_0 \alpha_1) Z^3 + (1 + \alpha_0 \alpha_1) Z^2 + \alpha_1 Z + \alpha_0
\]

\[
Z(Z^2 + \alpha_0) (Z^2 + \alpha_1)
\]

This transfer function exhibits 5 poles and 5 zeros, and for 0.1055.. < \alpha_0 < 1.0 and 0.5278.. < \alpha_1 < 1.0, the zeros are on the unit circle. Again, 5 poles and 5 zeros isn't a bad return for a filter formed with only two multiplications per output point, but there are still gains to be had.

Note the coefficients of this filter structure can be selected to obtain a Butterworth filter by placing all N roots at -1.0, [ie. \((Z+1)^N\)]. These roots can also be distributed over the stop band to obtain arbitrary out-of-band ripple structure including equiripple response.

The filter structure we are describing is a half-band filter, one with the 3-dB point at the quarter sample frequency with the added distinction of being formed from polynomials in \(Z^2\). In addition to using these filters as root-raised-cos matched filters, we can use them to reduce the bandwidth of the input signal by a factor of 2 and then reduce the sample rate by the same factor. In doing so we can take advantage of a remarkable relationship between resampling filters and resampling ratio known as the NOBLE IDENTITY [3]. This relationship, demonstrated in figure 4, states that a filter formed as a ratio of rational polynomials in \(Z^M\) followed by an M-to-1 downsampler maintains the same input-output relationship when the two operations (filtering and downsampling) are commuted subject to changing the polynomial indeterminate from \(Z^M\) to \(Z\).

After performing the noble identity substitution, the two-path filter is seen to have the structure indicated in figure 5. Examining the interaction of the single delay and the downsampling switches on the two-paths we immediately see that the combination of switches and delay is equivalent to the counter-clockwise rotating commutator indicated in figure 5.
In this configuration, the two paths distribute successive input data pairs so that path-0 processes the odd indexed input points while path-1 processes the even indexed points. This means that, when resampling, the work load per input data point is half that of the complete filter. For instance, if the two-path filter contains 4 all-pass stages, even though all 4 stages contribute to each output, only 2-stages are used to process any given input point. Hence the workload for a 4-stage, 9-pole filter, with the spectral response indicated in figure 6, is only 2 multiplications and 4.5 additions per input point. This is quite an impressive ratio of (non trivial) filter singularities per arithmetic operation.

5. TOTAL SYSTEM DESCRIPTION

The structure of the variable rate modem is shown in figure 7. To realize an arbitrary bandwidth, the cascade of resampling filters is preceded by an anti-aliasing filter and a programmable rate sampler based on a direct digital synthesizer (DDS).

![Diagram of Variable Rate Modem](image)

**FIGURE 7. VARIABLE RATE MODEM**

The bandwidth of the anti-aliasing filter is set by the highest bandwidth input signal, $F_{\text{max}}$. This in turn sets the lowest sample rate of the sampler, $F_{\text{smin}}$, which for the desired quarter sample rate relationship is $4 F_{\text{max}}$ Hz. The range of sample rates spans the interval $F_{\text{smin}}$ to $2 F_{\text{max}}$. An arbitrary sample rate, say $F_a$, can be obtained by establishing a sample rate satisfying (8)

$$F_{\text{smin}} < 2^M F_a < 2 F_{\text{smin}} \quad (8)$$

then filtering and decimating with a cascade of $M$ 2:1 resampling filters. These frequency relationships are demonstrated in figure 8.

![Diagram of Frequency Relationships](image)

**FIGURE 8. FREQUENCY RELATIONSHIPS OF MODEM**

To obtain 1-Hz (or better) resolution for all selected bandwidths we must have 4-Hz resolution at the decimated sample rate. Since all selected sampling frequencies are some power of 1/2 times the input sample rate, the programmable sampler is required to have a frequency resolution of 8-Hz.
As an example, if the bandwidth range of the modem is to extend to 1-MHz in 1-Hz steps the sampler must be programmable in 8-Hz step over the range 4 to 8 MHz. Continuing this example, the anti-aliasing filter passband edge is at 1-MHz with a transition bandwidth of 3-MHz. In addition, the first stage of the resampling filter must be capable of operating at the maximum input rate of 8-MHz while subsequent stages operate at appropriately lower rates.

A more detailed description of the filtering section of the modem is presented in figure 9. Here we see that the resampler is in fact a cascade of \( M \) stages of 2:1 resampling filters. These stages successively reduce the noise bandwidth and sample rate, in steps of one-half, till the selected data rate is realized. Each stage of the resampling filter is a two-path polyphase all-pass structure with each path operating at half the input rate of that stage. The final filter stage in the arrangement is the matched filter which accomplishes a 2:1 bandwidth reduction without a data rate reduction. Since no resampling is performed in this stage, two delays are required per all-pass structure as opposed to the single delay required in the earlier resampling stages. The difference in the two types of stages is emphasized in figure 10. The equalizer stages following the filter stages are all-pass filters similar to the structures used in the preceding two-path filters and are implemented as both single and dual delay options.

6. CONCLUSIONS

We have described an interesting application of two-path all-pass filters for communication systems. The filters perform the addressed tasks of root-raised-cos matched filtering and half bandwidth 2:1 resample filtering, as well as spectral phase equalization. The computational burden required to implement these filters is extremely low. Using these filters with a sampler which can be varied over an octave results in a modem with versatile variable bandwidth and sample rate options.

7. ACKNOWLEDGEMENT

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8. BIBLIOGRAPHY

