MULTI-COMPONENT SIGNAL ANALYSIS USING THE POLYNOMIAL TRANSFORM

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ABSTRACT

The Polynomial Transform (PT) and its discrete time counterpart, the Discrete Polynomial Transform (DPT), are recently developed analytical tools for estimation of continuous phase signals. In previous work we developed the basic theory and analyzed the performance of the PT/DPT for single component signals. In this paper we extend our results to signals consisting of multiple polynomial phase components. We then use the DPT to analyze sounds generated by the big brown bat, which produces complex polynomial phase signals to echo-locate objects.

1 INTRODUCTION

In some recent work we have developed the Polynomial Transform (PT) [1], and its discrete time counterpart, the Discrete Polynomial Transform (DPT) [2], [3]. The DPT (and PT) is a new tool for analyzing constant-amplitude continuous-phase signals. It is based on modeling the signal phase by a polynomial function of time (on a finite interval). The main properties of the DPT are its ability to identify the degree of the phase polynomial and to estimate its coefficients. The transform is robust to deviations from the ideal signal model, such as slowly-varying amplitude, additive noise and non-polynomial (but continuous) phase. Using the mathematical properties of the DPT we were able to derive computationally efficient estimation algorithms for continuous-phase signals. We were also able to show that the accuracy of the parameter estimates provided by the DPT are very close to the CRB [3], [4].

All of the work in [1] - [4] is for the case of a single polynomial phase signal. In this paper we develop an iterative procedure for estimating the parameters of multiple superimposed polynomial phase signals. The proposed algorithm is based on the DPT, and shares its computational simplicity. A preliminary statistical analysis of the algorithm has shown that it is able to provide accurate parameter estimates. This was validated by Monte-Carlo simulations.

The key property of the DPT is that, when applied to a single polynomial phase signal, it "collapses" the broadband signal to a single spectral line. The frequency of this line provides an estimate of the highest order polynomial phase parameter. By repeated application of the DPT followed by ‘phase unwrapping’ all the polynomial phase parameters are estimated.

In the case of multiple polynomial phase signals the DPT will have multiple spectral lines. By selecting one of these lines and proceeding as before, we obtain estimates of the polynomial phase parameters of one of the signals. A detection statistic is derived, and used to determine whether the selected lines correspond to a signal, or are due to an undesired cross-term.

Given the estimated parameters of one signal, it is possible to filter the data, effectively removing the signal. The procedure above can now be repeated on the filtered data, until all of the polynomial phase signals are removed. The estimated signal parameters can be used to reconstruct the component signals, to produce instantaneous frequency plots, etc.

We note that a different approach to this problem has been recently presented in [5] for the special case of ‘chirp’ signals (i.e., polynomial phase signals of second order).

The proposed algorithm has numerous applications to radar, sonar, and communications problems, as well as to the analysis of various natural signals. In this paper we apply the DPT to the analysis of signals emitted by the brown bat. We present an example to demonstrate the ability of the DPT to separately estimate the components of these complex signals.

In the next section we introduce the DPT. Motivated by the properties of the transform, we present an algorithm for estimating the parameters of a single polynomial-phase signal. In the third section we develop the criterion for detecting a single polynomial-phase signal. In the fourth section we present an iterative procedure for estimating the parameters of multiple superimposed polynomial phase signals. Finally, in the fifth section we present a numerical example.

2 ESTIMATING ONE SIGNAL

This section introduces the operator \( \text{DPT}_M[s(n), \tau] \) and the transform \( \text{DPT}_M^*[s(n), \omega, \tau] \). Motivated by
the properties of the transform, an algorithm for estimating the parameters of a polynomial-phase signal is derived. The approach used in this section is somewhat different than the approach used in [2].

We start by defining two linear operators. The forward $\tau$ shift operator is defined by

\[ \mathcal{F}_\tau \{ \phi(n) \} = \phi(n + \tau) \] (1)

and the $\tau$ difference operator is defined by

\[ \nabla_\tau \{ \phi(n) \} = \phi(n) - \phi(n + \tau) = [1 - \mathcal{F}_\tau] \{ \phi(n) \}. \] (2)

Successive operations are defined in the obvious way. Thus, the general forms are

\[ \mathcal{F}_\tau^k \{ \phi(n) \} = \phi(n + k\tau) \] (3)

and

\[ \nabla_\tau^k \{ \phi(n) \} = [1 - \mathcal{F}_\tau]^k \{ \phi(n) \}. \] (4)

The following theorem can be proven by induction.

**Theorem 1** Let

\[ \phi(n) = \sum_{m=0}^{M} a_m n^m. \] (5)

Then

\[ \nabla_\tau^{M-1} \{ \phi(n) \} = (-1)^{M-1} a_M \tau^{M-1} n + \gamma_M(\tau), \] (6)

where $\gamma_M(\tau)$ is a polynomial of order $M$ in $\tau$, but is not function of $n$.

Now, for the case

\[ s(n) = \exp\{j\phi(n)\} \] (7)

we have

\[ \exp \{ j \nabla_\tau^{M-1} \{ \phi(n) \} \} = \prod_{q=0}^{M-1} s^q(n + q\tau) \] (8)

where

\[ s^q(n) = \begin{cases} s(n), & \text{when } q \text{ is even}, \\ s^*(n), & \text{when } q \text{ is odd}. \end{cases} \] (9)

Motivated by equation (8) we define the operator

\[ DP_M[s(n), \tau] \] on $s(n)$ by

\[ DP_M[s(n), \tau] = \prod_{q=0}^{M-1} \left[ s^q(n + q\tau) \right]^{(M-1)} \] (10)

and the transform DPT of order $M$ of $s(n)$ as the Discrete Fourier Transform of $DP_M[s(n), \tau]$, i.e.,

\[ DPT_M[s(n), \omega, \tau] = \sum_n DP_M[s(n), \tau] \exp\{-j\omega n\Delta\}. \] (11)

From Theorem 1 and the definition of the DPT of order $M$ we conclude the following: Let $s(n)$ be a polynomial-phase signal of degree $M$. Then $DPT_M[s(n), \omega, \tau]$ has a global maximum at frequency

\[ \omega_0 = (-1)^{M-1} a_M \tau^{M-1}. \] (12)

The discussion above is only a brief introduction to the DPT. See [1]-[4] for more details. Next we present an algorithm for estimating the parameters of a polynomial-phase signal, using the DPT.

We assume that the signal is known to have polynomial phase of degree $M$ or smaller. The following algorithm estimates the coefficients $\{a_m\}$ sequentially, starting at the highest order coefficient $a_M$. At each step, except for the first one, the effect of the phase term of the highest order is removed. The algorithm proceeds as follows.

1) Substitute $m = M$ and $s^{(m)}(n) = s(n)$, $1 \leq n \leq N$.
2) Compute $\hat{a}_m$ by

\[ \hat{a}_m = \frac{(-1)^{m-1}}{m[(\Delta \tau)^{m-1} \arg \max_{\omega}(\|DPT_m[s^{(m)}(n), \omega, \tau]\|)}]}. \] (13)

3) Let

\[ s^{(m-1)}(n) = s^{(m)}(n) \exp\{-j \hat{a}_m(n\Delta)\}, \quad 1 \leq n \leq N. \] (14)

4) Substitute $m = m - 1$. If $m \geq 1$, go to step 2).

The choice of $\tau$ in step 2 will affect the accuracy of the estimated coefficients. Based on statistical analysis, we found that when the number of measurements and the signal-to-noise ratio tend to infinity, the optimal value of $\tau$ is $[N/(m+2)]$ when $m > 3$, and $[N/m]$ when $m \leq 3$.

The computation in step 2 can be done efficiently by using an FFT. The frequency which maximizes $\|DPT_m[s^{(m)}(n), \omega, \tau]\|$ is found by searching over the FFT frequency bins, followed by an interpolation step to refine the frequency estimate.

### 3 DETECTING ONE SIGNAL

In this section we consider the detection of constant-amplitude polynomial-phase signals. The problem is as follows: we are given $L = N_f - N_i + 1$ sampled of $z(n), \quad N_i \leq N_f$, consisting either of noise only, or of a constant-amplitude polynomial phase signal corrupted by noise. The noise process $w(n)$ is assumed to be a zero-mean white Gaussian noise with variance $\sigma^2$. In other words we have a binary hypothesis testing problem where

\[ H_1: \quad z(n) = s(n) + w(n), \quad n = N_i, N_i + 1, \ldots, N_f, \]

\[ H_0: \quad z(n) = w(n), \quad n = N_i, N_i + 1, \ldots, N_f, \]

where $z(n) = \tau(n) + y(n)$ and $s(n) = b_0(\mu(n) + y(n)) = b_0 \exp\{j\phi(n)\}$.
The probability density function of \( z(n) \) under each hypothesis is
\[
\begin{align*}
P(z(n)|H_1) &= \left( \frac{1}{\pi \sigma^2} \right)^L \exp \left\{ -\frac{(\sigma^2)^{-1}}{2} \sum_{n=N_t} \left[ (x(n) - b_0 u(n))^2 + (y(n) - b_0 v(n))^2 \right] \right\}, \\
&= \frac{1}{\pi \sigma^2} \left( \frac{1}{\pi \sigma^2} \right)^L \exp \left\{ -\frac{(\sigma^2)^{-1}}{2} \sum_{n=N_t} \left[ x^2(n) + y^2(n) \right] \right\}.
\end{align*}
\]
and
\[
\begin{align*}
P(z(n)|H_0) &= \left( \frac{1}{\pi \sigma^2} \right)^L \exp \left\{ -\frac{(\sigma^2)^{-1}}{2} \sum_{n=N_t} \left[ x^2(n) + y^2(n) \right] \right\}.
\end{align*}
\]

The generalized likelihood ratio detector for this problem is the ratio of the probability density functions of the measurements conditioned on the two hypotheses, with Maximum-Likelihood (ML) estimates of the unknown parameter substituted for the unknown parameter values.

\[
\Lambda(z(n)) = \frac{P(z(n)|H_1)_{H_1}}{P(z(n)|H_0)_{H_0}} \approx \frac{\eta}{\sigma^2 \log \eta}
\]

which can be written as
\[
\hat{b}_0 R \left\{ \sum_{n=N_t} \frac{1}{N_t} \exp \left\{ -j \phi(n) \right\} \right\} \approx 0.5 \left( \frac{\hat{b}_0^2}{\sigma^2} + 0.5 \log \eta \right).
\]

Next we note that the left hand side of (20) is \( L \) times \( \hat{b}_0^2 \). Thus
\[
\hat{b}_0^2 \geq \frac{\sigma^2}{L} \log \eta.
\]

The proposed procedure estimates the coefficients \( \{a_{m_p}\} \) sequentially, starting at the highest order coefficient of the strongest signal \( a_{M} \). The estimation process starts by picking the strongest spectral line of the DPT and using its frequency to estimate the highest order polynomial coefficient. Next, the effect of the phase term of the higher order is removed. This is continued until all the parameters of the candidate signal have been estimated. Then we compute the test statistic \( \hat{b}_0 \) as described in the previous section. The test statistic is compared to a threshold to determine whether the phase parameters we have estimated are the parameters of a signal component, or are a result of the spurious cross-term components produced by the non-linearity of the DPT. If the detection statistic exceeds the threshold, we accept the estimated parameters, and proceed to the estimation of the next component, as described below. If the detection statistic is below the threshold, we discard the estimated parameters, and repeat the process, starting with the second strongest spectral line. This process is repeated until a valid signal component is detected.

After the detection we remove this component from the composite signal, as follows:

(i) Multiply the data by \( \exp \left\{ -j \hat{\phi}(n\Delta) \right\} \) where \( \hat{\phi}(n\Delta) \) is the estimated phase of the signal.
(ii) Remove the low frequency components.
(iii) Multiply the result by \( \exp \left\{ j \hat{\phi}(n\Delta) \right\} \).

The procedure above can now be repeated on the remainder, until all of the polynomial-phase signal components are removed. The estimated signal parameters can be used to reconstruct the component signals. For example, one can estimate the instantaneous frequency (IF) of each signal as follows.
6 CONCLUSIONS

We presented an iterative algorithm for estimating the parameters of multiple superimposed polynomial phase signals. The algorithm is based on the ability of the DPT to estimate the parameters of a polynomial phase signal in the presence of other "interfering" signals. Having estimated the parameters of one of the signals, it can be filtered out from the composite signal. The procedure is then applied to the remainder, which contains a smaller number of components. In this paper we applied the DPT to the analysis of signal emitted by the big brown bat. The performance of the algorithm and its potential limitations are explored in some detail in [6].

References


5 AN EXAMPLE

In this section we demonstrate the DPT based algorithm for multiple signals by applying it to signals emitted by the big brown bat, *Eptesicus fuscus*. The bat uses sound to echo-locate objects and is able to navigate, detect, locate, classify and capture small insects in a highly cluttered, changing environment. The sounds generated by the brown bat can be described as pulses of multiple quadratic FM signals. The pulse repetition frequency and the signal parameters vary, depending on the task being performed by the bat, e.g., scanning, acquisition, tracking and terminal homing. The highly nonstationary and variable nature of the bat signals makes their analysis by conventional processing techniques quite difficult.

The following figures show the results of analyzing a short recording of the sounds emitted by a bat. Figure 1(a) depicts the sampled signal. The sampling frequency is 800 KHz. Figure 1(b) shows the spectrum of this signal after conversion to an analytical signal and decimation by a factor of 6. The signal is wideband, with most of its energy concentrated from 25 kHz to 110 kHz.

Figure 2(a) depicts the spectrum of $z_1(n)$ - the data after the removal of the first signal component. The estimated phase of the first component is $7.49 \times 10^4 + 3.25 \times 10^3 - 1.39$.

Figure 3(a) depicts the spectrum of $z_2(n)$ - the data after removal of the first two components. The estimated phase of the second component is $1.47 \times 10^5 + 6.47 \times 10^4 - 1.58$.

Figure 4 depicts the estimated instantaneous frequency versus time of the two components using the DPT based algorithm and the short-time Fourier Transform (STFT).

$\hat{\phi}_p(n\Delta) = \sum_{m=1}^{M} m\hat{a}_{m,p}(n\Delta)^{m-1}$. (25)

$2$ The data used in this example was provided by professor J. A. Simmons, Walter S. Hunter Laboratory of Psychology, Brown University, Providence, Rhode Island.
Figure 1: (a) The sampled bat signal. (b) The spectrum after decimation by 6.

Figure 3: (a) The spectrum used in the detection of the second component. (b) The spectrum after removing the second component.

Figure 2: (a) The spectrum used in the detection of the first component. (b) The spectrum after removing the first component.

Figure 4: The estimated instantaneous frequency versus time of the signal components. The solid lines correspond to the DPT based estimates, and the dots to estimates based on the STFT.