State-Space Characterization of Viterbi Detector Path Metric Differences

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Abstract

In the digital implementations of maximum-likelihood detectors based upon the Viterbi algorithm, bounds on the values of path metric differences are important parameters, and various techniques have been proposed for computing such bounds. This paper addresses the more general problem of characterizing the entire space of path metric differences, achievable from a given initial state, and calculating the probability density for the differences as a function of the distribution of the (noisy) channel output samples. Explicit results are given for two examples of interest in digital recording.

1 Introduction

The Viterbi algorithm, in its most general interpretation, provides a maximum-likelihood estimate of the state sequence of a Markov chain in the presence of additive noise. Since its introduction in the late 1960's, it has found wide use in communications, signal processing, and more recently, digital data recording. In all of these applications, considerations related to implementation and performance make it desirable to have bounds on the size of path metric differences. Two recent papers have addressed the problem of computing accurate bounds for these quantities [1],[2].

A broader, and more difficult problem, is to describe the entire state-space of path metric differences along with their steady-state probability distribution as a function of the statistics of the noise. This problem requires an understanding of the dynamics of the algorithm, which can be quite complex. When the solution is available, however, it can provide insights into such issues as optimal quantization of the path metric differences, and the impact of differing noise distributions on the performance of the Viterbi algorithm.

In this paper, we explicitly and completely characterize the state-space of path metric differences for two applications of the Viterbi algorithm taken from data communications and storage: the detection of binary, dicode partial-response and of Even-Mark-Modulation (EMM) trellis-coded duobinary (class-I) partial-response [2].

More precisely, we examine in detail the one-step dynamics of these two Viterbi detectors, assuming an appropriate range \( R \) of noisy output sample values, \( y \). With the all-0's state, denoted by \( x_0 \), as the initial state, we then determine the region \( P \) of states that can be reached from \( x_0 \) in one or more steps. This region will define the set of recurrent states of the system. It can also be characterized as the smallest region containing the state \( x_0 \) that is invariant under the one-step process represented by the detector algorithm. In other words, any other invariant region in the state space that contains \( x_0 \) must contain \( P \).

For the dicode channel, we also derive a closed-form expression for the difference metric distribution as a function of the noise statistics.

2 The Binary Dicode Channel

For background on the binary, dicode partial-response channel, we refer the reader to [2].

2.1 One-step dynamics

The trellis underlying the binary dicode channel is shown in Figure 1 a).

The corresponding difference metric algorithm may be found in [2]. As in [2], the noisy sample values \( y \) are assumed to lie in an interval \( R = [-r, r] \). Figure 2
III

\[ \frac{p(y)}{P(y)} \]

<table>
<thead>
<tr>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
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<tbody>
<tr>
<td>2</td>
<td>A</td>
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<td>C</td>
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</tr>
<tr>
<td>-5</td>
<td>-3</td>
<td>0</td>
<td>3</td>
<td>5</td>
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</tbody>
</table>

Figure 1: (a) Dicode trellis (b) EMM-coded duobinary.

shows regions of pairs \((a, y)\), where \(a\) denotes a difference metric value and \(y\) denotes a noisy sample value, that are relevant to describing the dynamics of the algorithm.

Table I describes the new difference metric \(a'\) determined by the input pairs \((a, y)\) to the algorithm.

From this representation, it is easily seen that the region \([-3,3]\) is invariant and, moreover, each state in that subinterval can be reached from the 0 state in one step. This implies that the region \(P = [-3,3]\) is the smallest invariant subspace containing the 0 state. This is consistent with the exact bounds proved in [2].

2.2 Steady-state distribution

We now incorporate the statistics of the noisy output samples and derive the steady-state distribution of the difference metric \(a\).

Let \(\rho(y)\) denote the probability density function (pdf) of the sample values and \(P(y)\) denote the corresponding cumulative distribution function (cdf). Similarly, let \(f(a)\) be the stationary distribution of the difference metric, with \(F(a)\) the corresponding cumulative distribution function. The difference metric algorithm translates directly into an equation involving these quantities:

\[
f(a) = \frac{1}{2} \rho(\frac{a - 1}{2}) \left[ 1 - F(a) \right] + \frac{1}{2} \rho(\frac{a + 1}{2}) F(a) + f(a) \int_{\frac{-3}{2}}^{\frac{3}{2}} \rho(y) dy.
\]

From this equation, we derive the general solution for \(F(a)\), stated in Proposition 1 below. Observe first that, by setting \(\rho(y) = \frac{1}{3}\) on the sample range \(R = [-2,2]\), we find that \(f(a) = \frac{1}{2}\), so it follows by symmetry that \(F(-3) = 0\) and \(F(3) = 1\).

**Proposition 1.** Let \(\rho(y), P(y), f(a), \) and \(F(a)\) be as above. Assume that \(\rho\) is continuous, and \(P(-2) = 0, P(2) = 1\). Then

\[
F(a) = \frac{P(\frac{a - 1}{2})}{1 - P(\frac{a + 1}{2}) + P(\frac{a - 1}{2})}
\]

for \(-3 \leq a \leq 3\).

**Proof of Proposition 1.**

Let \(t = F(a)\). Then, \(dt/da = F'(a) = f(a)\). Defining

\[
N(a) = 1 - \int_{\frac{-3}{2}}^{\frac{3}{2}} \rho(x) dx
\]

and

\[
M(a) = \frac{1}{2} \left[ \rho(\frac{a - 1}{2}) - \rho(\frac{a + 1}{2}) \right] - \frac{1}{2} \rho(\frac{-3}{2}),
\]

equation (1) becomes

\[
dtN(a) + dM(a) = 0.
\]

Since \(\partial N/\partial a = \partial M/\partial t\), this first order differential equation is exact. Therefore, it has the solution \(u + c\) with

\[
du = M da + N dt = (\partial u/\partial a) da + (\partial u/\partial t) dt
\]

Solving the equation in the conventional manner, and invoking the condition \(F(-3) = 0\), we obtain (2).

Note that if \(\rho\) is symmetric about \(y = 0\), then \(f\) is symmetric about \(a = 0\). We now apply equation (1) to three specific cases of interest.
Example 1. We have already seen that for the uniform density, \( p = \frac{1}{4} \) on \( R = [-2,2] \), the difference metric density is also uniform: \( f = \frac{1}{6} \) on \( P = [-3, 3] \).

Example 2. For the quadratic density, \( \rho(y) = \frac{3}{32}(4 - y^2) \) on \( R = [-2,2] \), the difference metric cdf is given by

\[
F(a) = \frac{1}{2} + \frac{a(45 - a^2)}{6(a^2 + 27)}
\]

Example 3. For the triangular symmetric density on \( R = [-2,2] \), the difference metric cdf is given by

\[
F(a) = \begin{cases} 
\frac{2a}{4} & \text{if } a \leq -1 \\
\frac{a^2 + 9}{2(a^2 + 9)} & \text{if } -1 < a \leq 1 \\
\frac{a^2 - 9}{4(a^2 + 9)} & \text{if } a \geq 1
\end{cases}
\]

The three cdf's are plotted in Figure 3.

![Figure 3: Cumulative distribution functions for uniform (solid), quadratic (dash), and triangular (dot-dash) inputs.](image)

3 EMM-coded Duobinary Channel

3.1 One-step dynamics

The detector trellis for the EMM-coded duobinary channel is shown in Figure 1 b). Of the three difference metrics \( DM(1,2) \), \( DM(2,3) \), and \( DM(1,3) \) any pair will determine the third. When the noisy sample region is again taken to be \( R = [-2,2] \), the LP bounds on the difference metrics are given by: \( DM(1,2) \in [-5,5] \), \( DM(2,3) \in [-10,6] \), \( DM(1,3) \in [-13,11] \). Computer simulation, along with intermediate results of the LP bound computations suggest the finer bounds \( DM(1,2) \in [-5,5] \), \( DM(2,3) \in [-10,6] \), \( DM(1,3) \in [-13,9] \). However, the RLP approach applied to these conjectured bounds does not confirm their validity: from the initial state with metrics \( DM(1,2) = 5 \), \( DM(2,3) = -8 \), \( DM(1,3) = -3 \), three consecutive noisy samples with value \( y = 2 \) lead to a state with \( DM(1,3) = 11 \). We will show in the next section that these improved bounds are indeed valid and, in fact, optimal.

For convenience, we parametrize the state-space in terms of pairs \( (a, b) \), where \( a = -DM(1,3) = DM(3,1) \) and \( b = DM(2,3) \). We refer again to Figure 2 to describe regions for the state parameter \( a \) and the noisy sample value \( y \). Table II describes the new state \((a', b')\) achieved from initial state \((a, b)\) as the sample value ranges over \( R \). The image of each state \((a, b)\) is a line segment, with the line segment from state \((u, v)\) to \((u', v')\) denoted by \([((u, v), (u', v'))\).

As we vary \((a, b)\), with \( a \) restricted to one of the five subregions, the one-step dynamics determine a polygonal image region whose boundary is specified by reference to Table II.

3.2 Recurrent region

We now give a complete characterization of the recurrent region \( P \) for this example, thereby confirming the validity and optimality of the improved difference metric bounds conjectured in the previous section.

We assume that \( a \) lies in the conjectured range \([-9,13]\). Figure 4 shows a region \( P \) of states \((a, b)\) that we will prove to be the recurrent state-space. To prove that this is indeed the recurrent region, we first must verify that the region is invariant under the Viterbi algorithm mapping. This can be done by referring to Table II. Figure 5 illustrates the sub-regions reached from states \((a, b)\) in \( P \) for each of the five zones for \( a \).

Next, we show that no proper subregion of \( P \) containing state \((0,0)\) is invariant under the mapping. This is accomplished by demonstrating that each state in \( P \) is reached from \((0,0)\) by a succession of iterations of the algorithm for some sequence of noisy samples in \( R \). In fact, we can show that seven iterations is sufficient to reach any state in \( P \).

Table III describes the succession of regions \( P_1, P_2, \ldots, P_7 \) reached after successive iterations start-
Conclusions

In this paper, we have explored issues related to determining the state-space of survivor path metric differences in applications of the Viterbi algorithm. For two cases of interest in digital recording, we described the one-step dynamics of Viterbi detectors, and characterized the recurrent region of difference metrics.

References


### TABLE I. One-step dynamics for dicode detector

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>(a, y) → 2y + 1</th>
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<tbody>
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<td>I</td>
<td>A</td>
<td>(a, y) → 2y - 1</td>
</tr>
<tr>
<td>II</td>
<td>B</td>
<td>(a, y) → a</td>
</tr>
<tr>
<td>III</td>
<td>A</td>
<td>(a, y) → 2y - 1</td>
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<td>III</td>
<td>B</td>
<td>(a, y) → a</td>
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<tr>
<td>III</td>
<td>C</td>
<td>(a, y) → 2y + 1</td>
</tr>
<tr>
<td>IV</td>
<td>B</td>
<td>(a, y) → a</td>
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<tr>
<td>IV</td>
<td>C</td>
<td>(a, y) → 2y + 1</td>
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### TABLE II. One-step dynamics for EMM/duobinary

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>(a, b) → (b - 3, -b), (b + 5, -b)</th>
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<td>(a, b) → (b + 5, -a - b - 5), (-a + b + 2, -b - 2)</td>
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<td>II</td>
<td>B</td>
<td>(a, b) → (b - a, -b)</td>
</tr>
<tr>
<td>III</td>
<td>A</td>
<td>(a, b) → (a + b + 8, -a - b - 5), (-a + b + 2, -b - 2)</td>
</tr>
<tr>
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<td>B</td>
<td>(a, b) → (-a + b + 2, -b - 2), (b - 3, -a - b + 3)</td>
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<tr>
<td>III</td>
<td>C</td>
<td>(a, b) → (b - 3, -b)</td>
</tr>
<tr>
<td>IV</td>
<td>B</td>
<td>(a, b) → (a + b + 8, -a - b + 3), (a + b + 8, -a - b - 5)</td>
</tr>
<tr>
<td>IV</td>
<td>C</td>
<td>(a, b) → (a + b + 8, -a - b + 3), (a + b + 8, -a - b - 5)</td>
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### TABLE III. Iterated image regions starting from (0,0)

<table>
<thead>
<tr>
<th></th>
<th>(0,0)</th>
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<th>(2,-2)</th>
<th>(8,-5)</th>
<th>(11,-8)</th>
<th>(-5,0)</th>
<th>(-6,5,15)</th>
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<td>$P_0$</td>
<td>(0,0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_1$</td>
<td>(-3,0)</td>
<td>(-2)</td>
<td>(0,0)</td>
<td>(3,0)</td>
<td>(11,-8)</td>
<td>(-5,0)</td>
<td>(-6,5,15)</td>
</tr>
<tr>
<td>$P_2$</td>
<td>(-5,5,25)</td>
<td>(-4,2)</td>
<td>(0,0)</td>
<td>(3,0)</td>
<td>(11,-8)</td>
<td>(-5,0)</td>
<td>(-6,5,15)</td>
</tr>
<tr>
<td>$P_3$</td>
<td>(-7,4)</td>
<td>(5,0)</td>
<td>(7,5,-2,5)</td>
<td>(7,25,-2,5)</td>
<td>(7,3)</td>
<td>(6,5,-3,5)</td>
<td>(11,-8)</td>
</tr>
<tr>
<td>$P_4$</td>
<td>(-9,4)</td>
<td>(1,2)</td>
<td>(5,0)</td>
<td>(9,-4)</td>
<td>(8,4)</td>
<td>(7,3,-4)</td>
<td>(13,-10)</td>
</tr>
<tr>
<td>$P_5$</td>
<td>(-8,5)</td>
<td>(2,-5)</td>
<td>(5,0)</td>
<td>(10,5)</td>
<td>(8,5,-5)</td>
<td>(13,-10)</td>
<td>(3,-5)</td>
</tr>
<tr>
<td>$P_6$</td>
<td>(-7,5)</td>
<td>(2,-5)</td>
<td>(5,0)</td>
<td>(10,5)</td>
<td>(8,5,-5)</td>
<td>(13,-10)</td>
<td>(3,-5)</td>
</tr>
<tr>
<td>$P_7$</td>
<td>(-6,5)</td>
<td>(1,2)</td>
<td>(5,0)</td>
<td>(10,5)</td>
<td>(8,5,-5)</td>
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