JOINT SYNCHRONIZATION AND DETECTION FROM SAMPLES TAKEN THROUGH INTEGRATE-AND-DUMP

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Abstract
We study the problem of joint synchronization and detection from samples taken by an integrate-and-dump filter, for both the optical Poisson and Gaussian channels. We show that the error probability due to timing error reduces to that with perfect synchronization as the sampling rate increases.

1 Introduction
The recent advent of high speed digital signal processing (DSP) chips has greatly motivated many studies on receiver algorithms which operate on sampled data instead of analog to produce the timing and symbol estimates [1,2,3,4,5,6,7,8]. In this paper, we study the problem of delay and sequence estimation for arbitrary baseband pulse shapes and arbitrary sampling rates as in [7] but in direct detection optical communications. We also study the same problem for the Gaussian channel with the samples taken not at the matched filter output but by an integrate-and-dump process.

The paper is organized as follows after the introduction. Section II has the derivation of the likelihood function for arbitrary baseband pulse shapes and sampling rates for optical Poisson systems and the same likelihood function using the integrate-and-dump filter for the Gaussian channel. The general results are then applied to several commonly used modulation schemes such as OOK and BPPM for the optical channel and the integral-and-dump filter for the Gaussian channel. Section III discusses the performance of the above described receivers both for optical and Gaussian channels. It contains the proof that the sequence error probability in the presence of timing error decreases monotonically to that with perfect synchronization as the sampling rate increases. Also, a lower bound on the error probability is derived for the Gaussian channel, and a comparison of this lower bound with the matched filter receiver is made between the matched filter receiver and the integrate-and-dump receiver for NRZ signaling. Section IV shows simulation results on the performance of the receiver described above for OOK and BPPM, and Section V concludes.

2 The Joint Likelihood Functions
2.1 The optical channel
In direct detection optical communications, for the popular modulation scheme PAM, an information sequence \( x = (x_0, x_1, \ldots, x_{N-1}) \), where \( x_k \) has \( Q \) levels, is transmitted by modulating the intensity of a laser. The intensity \( \lambda(t) \) of the corresponding received signal is

\[
\lambda(t) = \sum_{k=0}^{N-1} x_k p(t - kT - \tau') + \lambda_n, \tag{1}
\]

where \( \lambda_n \) is the noise intensity, \( \tau' \) is a timing error, \( T \) is the symbol duration and \( p(t) \) is an arbitrary baseband pulse of duration \( T \) seconds. The output of the photodetector is a doubly stochastic Poisson process \( N_t \), with intensity (photons/s) given by (1).

The receiver samples \( N_t \) every \( T_s = T/L \) seconds through integrate-and-dump to produce a sequence of samples \( K \),

\[
K = (K_{00}, K_{01}, \ldots, K_{0(L-1)}, K_{10}, K_{11}, \ldots, K_{1(L-1)}, \ldots, K_{(N-1)0}, K_{(N-1)1}, \ldots, K_{(N-1)(L-1)}).
\]

which are further processed to extract timing and symbol estimates. Mathematically these samples are given by

\[
K_{ij} = \int_{-\infty}^{\infty} g(t - iT - jT) dN_t, \tag{2}
\]

where

\[
g(t) = \begin{cases} 
1, & 0 < t \leq T_s, \\
0, & \text{otherwise}.
\end{cases}
\]

Because \( K \) is conditionally a Poisson random vector with independent components, the likelihood function for the joint estimation problem is

\[
l(K|x, \tau') = \prod_{i=0}^{N-1} \prod_{j=0}^{L-1} [K_{ij} \ln \lambda_{ij}(x, \tau') - \lambda_{ij}(x, \tau')], \tag{3}
\]

where

\[
\lambda_{ij}(x, \tau') = \lambda_n T_s + \lambda_k \sum_k x_k \tau' kT, \tag{4}
\]

because

\[
\lambda_{ij}(x, \tau') = \lambda_n T_s + \lambda_k \sum_k x_k \tau' kT.
\]
Actual Symbol Timing

\[
x_{i-1} \quad \tau \quad x_i \quad x_{i+1}
\]

\[\text{Start of Observations (i + 1)th sample} \]

**Figure 1: Timing Error of Unsynchronized Receiver**

\[
r_{ikj} = r[(i-k)T + jT_s - \tau],
\]

and

\[
r(t) = \int_{-\infty}^{\infty} g(\alpha)p(\alpha + t)d\alpha.
\]

It can be shown further that

\[
\lambda_{ij}(x, \tau) =
\begin{cases}
\lambda_n T_s + \lambda_s x_{i-1} T r_{i-1} t_{i-1}, & j < n, \\
\lambda_n T_s + \lambda_x x_{i-1} T r_{i-1} t_{i-1} + \lambda_s x_i r_i t_i, & j = n, \\
\lambda_n T_s + \lambda_s x_i r_i t_i, & n < j \leq L - 1,
\end{cases}
\]

where \( n = \lfloor \tau' / T_s \rfloor \). The above equation indicates that the received data in the \( j \)th transmission interval and \( i \)th sampling instant has a mean controlled by not only the \( z \)th symbol \( x_z \) but also the previous one, \( x_{z-1} \); further more, we can see the intersymbol interference (ISI) is only in the \( n \)th chip of duration \( T_s \). Apparently it will be reduced when \( T_s \) gets smaller, which is the intuitive idea of sampling more densely to get better performance. In Section III, we will prove theoretically that performance improves when the sampling rate increases.

If \( p(t) \) is a rectangular pulse and \( x_k \in \{0, 1\} \), then we have the popular modulation scheme OOK. In this case, \( r(t) \) is shown in Fig.2. For the special cases of one and two samples per symbol \( (L = 1, 2) \), when \( 0 \leq \tau' \leq T / 2 \), (3) reduces to

\[
\lambda(t) = \sum_{k=0}^{N-1} \sum_{j=0}^{Q-1} z_k t_0 \int_{-\infty}^{\infty} g(\alpha) p(\alpha + t) d\alpha.
\]

The likelihood function in this case is

\[
l(K|x, \tau') = \prod_{i=0}^{N-1} \prod_{j=0}^{Q-1} \prod_{m=0}^{L-1} [K_{ilm} \ln \lambda_{ilm}(x, \tau') - \lambda_{ilm}(x, \tau')].
\]

Then the demodulation is again to process the doubly stochastic Poisson process \( N_t \) by integrate-and-dump to obtain the photon counts or the samples \( K \) in time duration \( T_s \) seconds, \( T_s = T / Q L \), and \( Q L \) is the sampling rate for PPM. (See Fig.3.)

\[
K = (K_{000}, K_{001}, \ldots, K_{0Q-1}, K_{100}, K_{101}, \ldots, K_{1Q-1}, \ldots, K_{N-1,0}, K_{N-1,1}, \ldots, K_{N-1, Q-1, L-1}).
\]

The likelihood function in this case is

\[
l(K|x, \tau') = \prod_{i=0}^{N-1} \prod_{j=0}^{Q-1} \prod_{m=0}^{L-1} [K_{ilm} \ln \lambda_{ilm}(x, \tau') - \lambda_{ilm}(x, \tau')].
\]

where

\[
K_{ilm} = \int_{-\infty}^{\infty} g(t - iT - jT_s - mT_s) dN_t,
\]

\[
\lambda_{ilm}(x, \tau') = E[K_{ilm} | x, \tau'] = \sum_{k=0}^{N-1} \sum_{j=0}^{Q-1} z_{kj} r_{klm},
\]

\[
r_{klm} = r[(i-k)T + (j-j-M)L T_s + (m-n)T_s - \Delta],
\]

results are exactly the same as in [6] and [8], which are for \( L = 1 \) and \( L = 2 \) respectively.

For the pulse position modulation scheme, an information sequence \( x = (x_0, x_1, \ldots, x_{N-1}) \), where \( j = 0, 1, \ldots, Q - 1 \) and \( x_{kj} \) is one of the \( Q \) signals, is transmitted by modulating the intensity of the \( j \)th signal in the \( T \)-second symbol interval. If the \( j \)th signal is transmitted, then \( x_{kj} = 1 \), and \( x_{kj} = 0, j \neq i \). Each slot has duration \( T / Q \). The resultant intensity of the received signal can be mathematically expressed as

\[
\lambda_{il}(x, \tau') = \sum_{k=0}^{N-1} \sum_{j=0}^{Q-1} z_{kj} t_0 \int_{-\infty}^{\infty} g(\alpha) p(\alpha + t) d\alpha.
\]

The likelihood function in this case is

\[
l(K|x, \tau') = \prod_{i=0}^{N-1} \prod_{j=0}^{Q-1} \prod_{m=0}^{L-1} [K_{ilm} \ln \lambda_{ilm}(x, \tau') - \lambda_{ilm}(x, \tau')].
\]
and \( r' = M_T t + n T + \Delta_i \), \( 0 \leq \Delta_i \leq T_s \), \( n = \lfloor \frac{r' - MT}{Q} \rfloor \) for \( M = 0, 1, \ldots, Q - 1 \). The PPM scheme is more complicated than PAM in that it increases one more dimension in positioning during symbol interval \( T \).

The mean of the \( m^{th} \) sample has three cases, depending on \( l \) and \( M \). Which are:

1. When \( l = M + 1 \), \( 0 \leq M \leq Q - 2 \),
   \[
   E\{k_{im}|x, r'\} = \begin{cases} 
   z_{i-1} T_i, & m < n, \\
   z_{i-1} T_i, & m = n, \\
   z_{i-1} T_i, & m > n, 
   \end{cases}
   \]
   When \( l = 0 \), \( 0 \leq M \leq Q - 1 \),
   \[
   E\{k_{im}|x, r'\} = \begin{cases} 
   z_{i-1} T_i, & m < n, \\
   z_{i-1} T_i, & m > n, 
   \end{cases}
   \]
   When \( l = M + 1 \), \( 1 \leq M \leq Q - 1 \),
   \[
   E\{k_{im}|x, r'\} = \begin{cases} 
   z_{i-1} T_i, & m < n, \\
   z_{i-1} T_i, & m > n, 
   \end{cases}
   \]

It is observed that for PPM, we not only have intersymbol interference but also interslot interference. For example, as shown in Fig. 4 for \( Q = 2 \) and \( M = 0 \), we have both intersymbol and interslot interference, when \( l = 1 \), we have only interslot interference and so on. However, as \( T_s \) decreases or \( L \) increases, both intersymbol interference and interslot interference will reduce. If it is BPM, \( Q = 2 \) and assuming \( r' \leq r_i (M = 0) \), then

\[
\lambda_{im}(x, r') = \begin{cases} 
\lambda_z z_{i-1} T_i + \lambda_{i-1} T_i, & \text{if } n = 0 \text{ and } j = n, \\
\lambda_z z_{i-1} T_i + \lambda_{i-1} T_i, & \text{if } n = 0 \text{ and } j > n, \\
\lambda_z z_{i-1} T_i + \lambda_{i-1} T_i, & \text{if } n > 0 \text{ and } j = n, \\
\lambda_z z_{i-1} T_i + \lambda_{i-1} T_i, & \text{if } n > 0 \text{ and } j > n, 
\end{cases}
\]

and

\[
\lambda_{im}(x, r') = \begin{cases} 
\lambda_z z_{i-1} T_i + \lambda_{i-1} T_i, & \text{if } n = 0 \text{ and } j = n, \\
\lambda_z z_{i-1} T_i + \lambda_{i-1} T_i, & \text{if } n = 0 \text{ and } j > n, \\
\lambda_z z_{i-1} T_i + \lambda_{i-1} T_i, & \text{if } n > 0 \text{ and } j = n, \\
\lambda_z z_{i-1} T_i + \lambda_{i-1} T_i, & \text{if } n > 0 \text{ and } j > n, 
\end{cases}
\]

which is the same as in [6].

2.2 The Gaussian channel

The output samples of the integrate-and-dump receiver for the Gaussian channel are:

\[
R_{ij} = \sum_k z_k \int_{-\infty}^{\infty} g(\alpha) p(\alpha + (i - k) T + j T_s - r') d\alpha + N_{ij},
\]

where \( r' \) and \( g(\alpha) \) have the same definitions as for the optical channel, and

\[
R = (R_{00}, R_{01}, \ldots, R_{0(L-1)}, \ldots, R_{N-1} R_{N-1})
\]

is a Gaussian random vector with the components independent of one another, with variance \( N_0/2L \) and mean

\[
\mu_{ij}(x, r') = \begin{cases} 
\lambda_z z_{i-1} T_i, & j = n, \\
\lambda_z z_{i-1} T_i, & j > n, 
\end{cases}
\]

Then the likelihood function is

\[
l(x | r') = \sum_{i=0}^{N-1} \sum_{j=0}^{L-1} |R_{ij} \mu_{ij}(x, r') - \frac{1}{2} \nabla^2_{ij}(x, r')|.
\]

3 Performance

3.1 The optical channel

We first show that the error probability \( Pr(x, y | r') \) of deciding \( y \) while actually \( x \) is sent, using the receiver described in Section II, will theoretically reduce to the
same without the timing error as the sampling rate \( L \) goes to infinity.

Let \( l_x \) be the likelihood function without timing error \((\tau = 0)\) of sequence \( z \) with length \( N \), and \( l_y \) the one with \( \tau \neq 0 \); assume that \( y_{-1} = z_{-1} \) and 
\[ y_N = z_N. \]

Then
\[
Pr(x, y|\tau) = Pr(l_y - l_x \geq 0) = 
Pr\left( \sum_{i=0}^{N-1} \sum_{j=0}^{L-1} (K_{ij} \ln \frac{\lambda_{ij}(y, \tau')}{\lambda_{ij}(x, \tau')}) - [\lambda_{ij}(y, \tau') - \lambda_{ij}(x, \tau')] \geq 0 \right) = 
Pr \left[ \sum_{i=0}^{N-1} \sum_{j=0}^{L-1} \lambda_{ij}(y, \tau') - \lambda_{ij}(x, \tau') \geq 0 \right] 
\geq \sum_{i=0}^{N-1} \sum_{j=0}^{L-1} \Delta \lambda_{ij}(\tau') \right], \quad (14)
\]

while
\[
Pr(x, y|\tau') = Pr(l_y - l_x \geq 0) = 
Pr\left( \sum_{i=0}^{N-1} \sum_{j=0}^{L-1} (K_{ij} \ln \frac{\lambda_{ij}(y, \tau)}{\lambda_{ij}(x, \tau)} - [\lambda_{ij}(y, \tau) - \lambda_{ij}(x, \tau)] \geq 0 \right) = 
Pr \left[ \sum_{i=0}^{N-1} \sum_{j=0}^{L-1} \Delta \lambda_{ij}(\tau') \right]. \quad (15)
\]

From Fig.1, note that
\[
\begin{align*}
K_{(i+1)j} &= K_{ij}^*, & j = 0, \ldots, n - 1, \\
\lambda_{(i+1)j} &= \lambda_{ij}^*, & j = 0, \ldots, n - 1,
\end{align*}
\]

After some simplification, we have
\[
Pr(x, y|\tau) = 
Pr\left( \sum_{i=0}^{N-1} \sum_{j=0}^{L-1} [K_{ij} \ln \frac{\lambda_{ij}(y, \tau)}{\lambda_{ij}(x, \tau)} + K_{in} \ln \frac{\lambda_{in}(y, \tau)}{\lambda_{in}(x, \tau)}] \geq \sum_{i=0}^{N-1} \sum_{j=0}^{L-1} \Delta \lambda_{ij}(\tau')] \right), \quad (16)
\]

As \( T_s \to 0 \) or equivalently \( L \to \infty \), it can be easily shown that
\[
\lim_{L \to \infty} \Delta \lambda_{in} = 0. \quad (17)
\]

\[
\lim_{L \to \infty} Pr(K_{in} = 0) = 1. \quad (18)
\]

\[
\lim_{L \to \infty} Pr\left( \sum_{j \neq n} K_{ij} \ln \frac{\lambda_{ij}(y)}{\lambda_{ij}(x)} \right) = \Pr\left( \sum_{j \neq n} K_{ij} \ln \frac{\lambda_{ij}(y)}{\lambda_{ij}(x)} \right).
\]

\[
\lim_{L \to \infty} \sum_{j \neq n} \Delta \lambda_{ij}(\tau) = \sum_{j \neq n} \Delta \lambda_{ij}. \quad (19)
\]

Hence
\[
\lim_{L \to \infty} Pr(x, y|\tau') = Pr(x, y|\tau = 0). \quad (20)
\]

### 3.2 The Gaussian Channel

The performance in the Gaussian channel is much easier to analyze, because the difference \( \Delta \lambda \) between \( l_x \) and \( l_y \) is also a Gaussian r.v. It can be shown that the lower bound on the error probability is
\[
Pr(x, y|\tau) \geq Pr(K_{in} \geq 0) = \frac{1}{2} \text{erfc} \left( \sqrt{\frac{LE_b \nu_m}{2N_0}} \right),
\]
where \( \nu_m \) is the mean of Gaussian r.v. \( \Delta \lambda_{in} = l_x - l_y \) with \( x \) and \( y \) differ only in the \( m \)th position.

If \( p(t) \) is a rectangular (NRZ) pulse, the lower bound with arbitrary sampling rate \( L \) is
\[
Pr(x, y|\tau) \geq \frac{1}{2} \text{erfc} \left( \sqrt{\frac{E_b}{N_0}\left[1 - \frac{2T}{T} + 2L\left(\frac{\tau^2}{T^2}\right)\right]} \right), \quad (21)
\]
which is the same lower bound from the output samples of the matched filter receiver in [7]. This implies that the integrate-and-dump receiver described above can potentially perform as well as the matched filter receiver (more complicated) which has correlated samples for NRZ signaling.

### 4 Simulation Results

In the simulation, the maximization of the likelihood function is implemented efficiently through a dynamic programming (Viterbi) algorithm \( r' (M=16) \), using the following recursive formula:
\[
l_i = l_{i-1} + \sum_{j=0}^{L-1} \left[ K_{ij} \ln (1 + \frac{\mu_{ij}(x, \tau') - \mu_{ij}(x, \tau')}{\lambda_{in} T_s}) \right]. \quad (22)
\]

The performance of the receiver with OOK signaling and BPPM is obtained by running 100,000 independent sequences of length \( N = 20 \) and is shown in Fig. 5. and Fig.6., which indicate that the bigger the sampling rate, the better the performance is, as expected.
5 Conclusion

The problem of sequence and timing error estimation for arbitrary baseband pulse shapes and arbitrary sampling rates in integrate-and-dump optical communications has been studied. It is shown that the error probability due to timing error reduces to the same with perfect synchronization as the sampling rate goes to infinity and a lower bound on the error probability is also derived. The general results are then applied to the popular cases of OOK and BPPM. We have also studied the same problem for the Gaussian channel with samples taken by an integrate-and-dump filter. A lower bound on the error probability is also derived, and it is shown to be the same as the one with samples taken from the matched filter for NRZ signaling.

References