Optimum Active Array Shape Calibration

Boon C. Ng

Defense Science Organization

20, Science Park Drive, Singapore 0511, Republic of Singapore.

Arye Nehorai*

Department of Electrical Engineering
Yale University
New Haven, CT 06520, USA.

Abstract

This paper examines several array shape calibration algorithms based on maximum likelihood and eigenstructure methods using known calibration source signals. A compact expression for the corresponding Cramér-Rao bound (CRB) on the sensor location parameters is presented. For uncorrelated calibrating sources, a necessary and sufficient condition for the optimal choice of bearings that minimize the CRB is derived. Asymptotic covariances of the estimation errors for the calibration algorithms are also presented and compared with the CRB.

1 Introduction

In the problem of multiple source location, measurements are usually taken from spatially distributed sensors and processed to determine the direction of arrivals of source signals impinging on the array of sensors. In most modern array processing algorithms [1, 2, 8, 12], the array manifold are assumed to be known analytically or otherwise calibrated and stored in a table. It is clear that this would require the sensor locations to be known precisely. In practice, it is rare that sensor arrays (e.g. hydrophones or antennas) can be deployed with the sensor positions known exactly.

In the presence of sensor position errors, the performances of these array processing algorithms will degrade. There are two approaches to this problem. The first approach uses self-calibrating techniques [3, 4] which do not require calibrating sources. Although attractive, such techniques usually require the knowledge of the location of one sensor and the direction to a second sensor. Furthermore, they suffer from ambiguity problems when the sensors are collinear. The second approach uses calibrating sources with known bearings [6, 7]. This paper adopts the second approach and assumes that the waveforms of the calibrating sources are known. This added assumption is not unreasonable, since in practice there is usually some control over the kind of signals that are used for these calibrating sources. Furthermore, the performance is expected to be better than the case with unknown signal waveforms. We will call this the active array shape calibration problem.

In this paper, we will present the Cramér-Rao bound (CRB) on the sensor position errors for the active array shape calibration problem in a compact form. The CRB will be used to reveal information on how to select the number of sources and choose the source bearings so that the best performance can be achieved. We will present Newton-type algorithms for the maximum likelihood estimator as well as eigenstructure-based methods such as MUSIC [1] and MODE [8]. Asymptotic covariances for the ML and eigenstructure-based methods will also be presented.

2 The Data Model

Consider an array of m omnidirectional sensing elements in a 2-D plane with n far-field narrowband sources at center frequency $f_o$. The received signal vector can be written as

$$y(t) = A(\omega)s(t) + e(t)$$

(1)

where

$$\omega = [x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m]^T$$

(2)
The $i^{th}$ column of $A(\omega)$ is given by $a(\omega, \theta_i)$ where

$$a(\omega, \theta_i) = [e^{2\pi f s_1 \tau_1(\theta_i)}, e^{2\pi f s_2 \tau_2(\theta_i)}, \ldots, e^{2\pi f s_N \tau_N(\theta_i)}]^T$$

and $\tau_k(\theta_i)$ is the propagation delay given by

$$\tau_k(\theta_i) = [\sin \theta_i, \cos \theta_i][x_k, y_k]^T$$

where $(x_k, y_k)$ is the position of the $k^{th}$ sensor and $\theta_i$ is the bearing of the $i^{th}$ calibration source.

In the problem of active array shape calibration, the objective is to estimate the position of the sensors with the assumption that the source bearings and waveforms are known exactly, given $y(t)$, $t = 1, 2, \ldots, N$. Since $y(t)$ has both deterministic and stochastic components, it is convenient to define the generalized expectation operator [15]

$$E(\cdot) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} E(\cdot)$$

where $E$ is the usual expectation operator with respect to the stochastic components. Assuming $s(t)$ satisfies regularity conditions [5, 9] so that the limit of its sample covariance exists, then

$$E_s(t)s^*(u) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} s(t)s^*(u) = P \delta_{t,u}$$

$$E_s(t)s^T(u) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} s(t)s^T(u) = 0 \forall t, u$$

The noise is assumed to be Gaussian distributed, with zero mean and

$$E_n(t)n^*(u) = \sigma^2 P \delta_{t,u}$$

$$E_n(t)n^T(u) = 0 \forall t, u$$

Note that $y(t) \sim \mathcal{G}(A(\omega)s(t), \sigma^2 I)$.

### 3 Cramér-Rao Bound

The Cramér-Rao bound provides a lower bound on the variance of the sensor position errors of any unbiased estimator. It will be useful to investigate how the number of sources and choice of bearings affect this bound. Let $\alpha$ be the deterministic $(2m + 1) \times 1$ parameter vector given by

$$\alpha = [\sigma^2, \omega]^T$$

where $\omega$ is defined in (2).

**Theorem 3.1** Under the assumption of the signal model in (6)-(9), the CRB covariance matrix for $\alpha$ is given by

$$C_{\alpha(\alpha)} = \begin{bmatrix} \frac{\sigma^4}{m^2N} & 0 & 0 \\ 0 & C_{xx} & C_{xy} \\ 0 & C_{yx} & C_{yy} \end{bmatrix}$$

where

$$C_{xx} = \frac{\sigma^2}{2N} \left( A_x P_N A_x^* \otimes I - Re(A_x P_N A_y^* \otimes I) \right)$$

$$C_{yy} = \frac{\sigma^2}{2N} \left( A_x P_N A_x^* \otimes I - Re(A_x P_N A_y^* \otimes I) \right)^{-1}$$

$$C_{xy} = C_{yx}^T = \left( I - Re(A_x P_N A_y^* \otimes I) \right)^{-1}$$

$$A_x = \frac{2\pi}{\lambda_o} A \Lambda_{xin}$$

$$A_y = \frac{2\pi}{\lambda_o} A \Lambda_{cos}$$

$$P_N = \frac{1}{N} \sum_{t=1}^{N} s(t)s^T(t)$$

$$\Lambda_{xin} = \begin{bmatrix} \sin \theta_1 & 0 \\ 0 & \sin \theta_n \end{bmatrix}$$

$$\Lambda_{cos} = \begin{bmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_n \end{bmatrix}$$

$\lambda_o$ is the wavelength, and $diag(A) = \text{column vector formed from the diagonal of } A$.

**Proof:** See [16].

To see how the CRB depends on the characteristis of the calibrating sources, let

$$\sigma_{ij} = \frac{2\pi}{\lambda_o} (x_i \sin \theta_i + y_i \cos \theta_i)$$

and

$$p_{kl} = [P_N]_{kl}$$

894
The $ii^{th}$ element of $C_{xx}$ and $C_{yy}$ can be shown to be

$$
[C_{xx}]_{ii} = \frac{\sigma^2 \lambda^2}{8N\pi^2} \left( \sum_{k,l} p_{kl} \sin \theta_k \sin \theta_l e^{j(\phi_{kx} - \phi_{lx})} \right)
$$

$$
Re \left( \frac{\sum_{k,l} p_{kl} \sin \theta_k \cos \theta_l e^{j(\phi_{ky} - \phi_{ly})}}{\sum_{k,l} p_{kl} \cos \theta_k \cos \theta_l e^{j(\phi_{klx} - \phi_{klx})}} \right)^2
$$

$$
[C_{yy}]_{ii} = \frac{\sigma^2 \lambda^2}{8N\pi^2} \left( \sum_{k,l} p_{kl} \cos \theta_k \cos \theta_l e^{j(\phi_{kx} - \phi_{lx})} \right)
$$

$$
Re \left( \frac{\sum_{k,l} p_{kl} \sin \theta_k \sin \theta_l e^{j(\phi_{ky} - \phi_{ly})}}{\sum_{k,l} p_{kl} \cos \theta_k \cos \theta_l e^{j(\phi_{klx} - \phi_{klx})}} \right)^2
$$

Using (22)-(23), (24) can be written as

$$
V(\theta) = \frac{\lambda^2 \sigma^2 n}{8N\pi^2} \left( \sum_{k=1}^{n} \sin^2 \theta_k \cos^2 \theta_l \right)^{-1} \left( \frac{\sum_{k=1}^{n} \sin \theta_k \cos \theta_l}{\sum_{k=1}^{n} \sin \theta_k \cos \theta_l} \right)^2
$$

Theorem 3.2 For uncorrelated calibrating sources, the CRBs in both the $x$ and $y$ coordinates are minimized if and only if the bearings of the sources satisfies

$$
\sum_{k=1}^{n} e^{j2\theta_k} = 0
$$

The minimum value of the $V(\theta)$ is given by

$$
V_{\text{min}}(\theta_{\text{opt}}) = \frac{\sigma^2 \lambda^2}{2Nn\pi^2}
$$

Proof: Minimizing $V(\theta)$, is equivalent to maximizing

$$
\sum_{k=1}^{n} \sum_{l=1}^{n} \sin^2 \theta_k \cos^2 \theta_l - \left( \sum_{k=1}^{n} \sin \theta_k \cos \theta_l \right)^2
$$

$$
= \frac{n^2}{4} - \frac{1}{4} \sum_{k=1}^{n} \sum_{l=1}^{n} \cos(2\theta_k - 2\theta_l)
$$

$$
= \frac{n^2}{4} - \frac{1}{4} \sum_{k=1}^{n} \sum_{l=1}^{n} e^{j2(\theta_k - \theta_l)}
$$

$$
= \frac{n^2}{4} - \frac{1}{4} \sum_{k=1}^{n} e^{j2\theta_k}
$$

The results of (26)-(27) follows easily from direct substitution \(\Box\).

Corollary: A sufficient condition for minimizing the CRB is that the bearings are uniformly distributed.

Remark: Note that the minimum CRB varies uniformly as the reciprocal of the number of sources as well as the number of snapshots. As an example, for $n = 2$, the uncorrelated sources can be selected to be distributed such that they are spatially separated by $\frac{\pi}{2}$; for $n = 3$, the source separation should be $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ and so on.

4 MLE and Eigenstructure-based Calibration Algorithm

Assuming the nominal positions of the sensor to be reasonably close to the true locations, Newton-type algorithms can be devised for calibration algorithms based on maximum likelihood (ML) and eigenstructure-based (ES) estimators.

The ML calibration algorithm for the sensor position is based on the estimates that maximizes the likelihood function of the observed data. That is,

$$
\omega_{\text{ML}} = \arg\max_{\omega} \left\{ (\pi \sigma^2)^{-N} \exp \left( -\frac{1}{\sigma^2} \sum_{t=1}^{N} \|y(t) - A(\omega)s(t)\|^2 \right) \right\}
$$

$$
= \arg\min_{\omega} Tr(A(\omega)P_N A^*(\omega) - 2R(\omega)P_{\text{ref}})
$$

$$
= \arg\min_{\omega} f_{\text{ML}}(\omega)
$$

where

$$
P_{\text{ref}} = \frac{1}{N} \sum_{t=1}^{N} s(t)y^*(t)
$$

$$
f_{\text{ML}}(\omega) = Tr(A(\omega)P_N A^*(\omega) - 2R(\omega)P_{\text{ref}})
$$

The approximate gradient and Hessian of the ML cost function in (29) is given by (see [16])

$$
f'_{\text{ML}}(\omega) = 2R \text{ diag} \left[ A_r \Gamma, 0, 0, A_y \Gamma \right]
$$
where $r = P_N A^*(\omega) - P_8$.

The ES calibration algorithms are based on the class of subspace-based array processing algorithms [8, 9, 12]. The cost function of this class of algorithms have the form

$$f_{es}(\omega) = Tr \left( A^*(\omega) \tilde{G} \tilde{G}^* A(\omega) W \right)$$

(35)

where $W$ is a positive definite weighting matrix and $\tilde{G}$ is a finite sample estimate of $G$, the matrix of noise eigenvectors. In our case, the approximate gradient and Hessian of the ES cost function in (35) is given by (see [16])

$$f'_e(\omega) = 2Re \begin{bmatrix} \tilde{G}^* A W A^*_y & 0 \\ 0 & \tilde{G}^* A W A^*_y \end{bmatrix}$$

(36)

and

$$f''_e(\omega) = 2Re \begin{bmatrix} A^*_x A W A^*_y & A^*_x A W A^*_x \\ A^*_y A W A^*_x & A^*_y A W A^*_y \end{bmatrix} \odot \begin{bmatrix} \tilde{G}^* \tilde{G}^* \\ \tilde{G}^* \tilde{G}^* \end{bmatrix}$$

(37)

Using the above results, iterative solutions to solve for the different estimates can be implemented by the following control equations:

$$\omega_{k+1} = \omega_k + \beta_k [f''(\omega_k)]^{-1} f'(\omega_k)$$

(38)

The step length $\beta_k$ in the search direction is chosen to be the largest number in the sequence $\{1, \frac{1}{2}, \frac{1}{4}, \ldots\}$ that satisfies the inequality $cost(\omega_k) - cost(\omega_{k+1}) > 0$ [11].

Remark: For $m$ sensors, the ML algorithm requires $n \geq 2$ while the ES type of algorithms require $m - 1 \geq n \geq 2$. The dimension of the noise subspace set the upper limit for the latter case.

5 Asymptotic Results of Calibration Algorithms

The asymptotic covariance matrices of the calibration algorithms will be given here. The details can be found in [16].

R1: The covariances of the ML estimator and the CRB are asymptotically the same, that is

$$C''_{mlr}(\omega) = C''_{crb}(\omega)$$

(39)

where $C''_{mlr}(\omega)$ is the lower block diagonal matrix in (11)-(19) with $P_N$ replaced by $P$. This result is not surprising as the MLE is known to be asymptotically efficient when the likelihood function is regular (which is satisfied when the source signal waveforms are known).

R2: The asymptotic covariance matrix of the ES estimator is given by

$$C''_{es}(\omega) = \frac{\sigma^2}{2N} (Re(D \odot E))^{-1} Re(D \odot F) (Re(D \odot E))^{-1}$$

(40)

where

$$D = \begin{bmatrix} GG^* & GG^* \\ GG^* & GG^* \end{bmatrix}$$

(41)

$$E = \begin{bmatrix} A_x A^*_x & A_x A^*_x \\ A_y A^*_y & A_y A^*_y \end{bmatrix}^T$$

(42)

$$F = \begin{bmatrix} A_x (A^* U A^*) A_x (A^* U A^*) \\ A_y (A^* U A^*) A_y (A^* U A^*) \end{bmatrix}^T$$

(43)

$$U = S A_S (A_S - \sigma^2 I)^{-2} S$$

(44)

Here $S$ is the signal eigenvector matrix and $A_S$ is the diagonal matrix of the largest eigenvalues.

MUSIC: The asymptotic covariance of the MUSIC estimator (with $W = I$) is given by the expression

$$C''_{music}(\omega) = \frac{\sigma^2}{2N} (Re(D \odot E_{music}))^{-1} Re(D \odot F_{music}) . (Re(D \odot E_{music}))^{-1}$$

(45)

where

$$E_{music} = \begin{bmatrix} A_x A^*_x & A_x A^*_y \\ A_y A^*_y & A_y A^*_x \end{bmatrix}^T$$

(46)

$$F_{music} = \begin{bmatrix} A_x (A^* U A^*) A_x (A^* U A^*) \\ A_y (A^* U A^*) A_y (A^* U A^*) \end{bmatrix}^T$$

(47)

MODE: The asymptotic covariance of MODE (with $W = (A^* U A)^{-1}$) is given by the expression

$$C''_{mode}(\omega) = \frac{\sigma^2}{2N} (Re(D \odot E_{mode}))^{-1}$$

(48)

where

$$E_{mode} = \begin{bmatrix} A_x (A^* U A)^{-1} A_x & A_x (A^* U A)^{-1} A_y \\ A_y (A^* U A)^{-1} A_x & A_y (A^* U A)^{-1} A_y \end{bmatrix}^T$$

(49)

R3: The asymptotic covariance of the class of eigenstructure based estimators satisfy the following relations:

$$C''_{est}(\omega) \geq C''_{mlr}(\omega) = C''_{crb}(\omega)$$

(50)
The proofs of the assertions R2-R3 can be found in [16]. In general, estimators based on the sample covariance matrix (which includes the class of eigenstructure-based estimators) for the deterministic signal model is not asymptotically efficient. This is because the sample covariance matrix is not a sufficient statistic for \( \omega \).

6 Summary and Conclusions

Active array shape calibration using the maximum likelihood method and eigenstructure based method were presented. The following results were established:

- The Cramér-Rao bound of the calibration problem.
- A necessary and sufficient condition for the bearings of uncorrelated calibrating sources to achieve minimum variance of the sensor position estimates.
- Newton-type calibration algorithms based on maximum likelihood and eigenstructure of the sample covariance matrix.
- The asymptotic covariances of the estimators.
- The maximum likelihood calibration method is asymptotically efficient.

References


