Wavelet Transform Mother Mapper Operator

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Abstract
The wavelet transform domain characterization of a signal is highly dependent upon the mother wavelet which is chosen as the kernel of the transform. A new operator, the Mother Mapper Operator, is derived to efficiently evaluate several transforms of the same signal with different mother wavelets. The Mother Mapper Operator directly maps a transform with respect to one mother wavelet to a new transform with respect to a new, different mother wavelet. The Mother Mapper Operator leads to signal representations which involve multiple mother wavelets instead of just one mother wavelet.

Introduction
Wavelet theory is a functional decomposition analogous to Fourier theory. Unlike a Fourier transform though, the wavelet transform simultaneously localizes in two variables, scale and translation; the Fourier transform localizes only in frequency. The wavelet transform has several advantages over Fourier transforms for wideband and/or nonstationary processes; it may also be applied in either the spatial or temporal domains [1-9, 11, 12]. Each wavelet transform is parameterized by a mother wavelet, \( g(x) \). When the mother wavelet is changed the characteristics of the wavelet transform change. The same signal may have very different representations when different mother wavelets are employed.

In Fourier analysis the only signal which can be used for decomposing a signal is a tone; it is the kernel of the Fourier integral transformation. With wavelet transforms the mother wavelet is the kernel of the integral transform. The freedom for choosing this mother wavelet may be used to great advantage in characterizing and compressing signals. A wavelet transform of a signal does not have much meaning unless the mother wavelet is explicitly stated (unlike a Fourier transform whose kernel is known to be a tone).

Suppose a signal, \( f(x) \), is represented by a wavelet transform with respect to a particular mother wavelet, \( g(x) \), and that this transform domain representation has compact support. Now suppose a second mother wavelet, \( g_2(x) \), leads to a transform domain representation with spread out support (noncompact). Obviously, if the information in \( f(x) \) is to be saved or transmitted, then the compact representation (provided by the mother wavelet \( g(x) \)) will lead to efficiencies. Thus many different mother wavelets may be used to create an efficient representation.

For the purpose of efficiently evaluating wavelet transforms with respect to several, different mother wavelets a new operator, the Mother Mapper Operator, is presented. This operator will allow several mother wavelets to be considered without degrading the efficiency of the compression (or other) processor. The operator can also be applied to wideband cross ambiguity function generation to enhance the efficiency and structure of the processor [13].

Wavelet Transforms
Before defining the wavelet transform, admissible functions are defined. An \( L^2 \) function, \( g \), is an admissible function if:

\[
C_g \int_{\mathbb{R}} \frac{|G(\omega)|^2}{\omega} \, d\omega < \infty
\]

where \( G(\omega) \) is the Fourier transform of \( g \). This condition essentially states that the mother wavelets must be "bandpass" processes.

The one-dimensional wavelet transform operator, \( \mathcal{W}_g \), maps an \( L^2 \) (square integrable in its variable) signal which is real or complex as follows:

\[
\mathcal{W}_g : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R} \setminus \{0\} \times \mathbb{R}).
\]

The wavelet transform of a function, \( f \), with respect to a given
an admissible mother wavelet, g, exists and is defined as:

\[(W_g f)(a, b) = |a|^{-\frac{1}{2}} \int f(x) g^*(x-b) \frac{dx}{a}\]

\[= \langle f, g_{a,b} \rangle = \langle f, U(a, b) g \rangle\]

where superscript "*" denotes complex conjugation and \(\langle \cdot, \cdot \rangle\) denotes an \(L^2\) inner product on the Hilbert space which contains f and g as members. Note that this definition requires that \(g(x)\) is an admissible function. The \(W_g f(a, b)\)'s are the wavelet transform coefficients. \(g_{a,b}\) is defined by a unitary affine mapping

\[U(a, b) : g(x) \mapsto \frac{1}{\sqrt{a}} g\left(\frac{x-b}{a}\right)\]

The \(g_{a,b}\) functions are the time scaled and time translated versions of the mother wavelet, \(g(x)\), where parameter "a" denotes the time scale and parameter "b" represents a time translation. \(g_{a,b}(x)\) is chosen to have a \(c_g\) value of unity so that the wavelet transform is isometric ("energy" is the same in each domain) and that any coefficient for a particular \((a, b)\) may be "fairly" compared or combined even across a huge range of scales.

Note that for continuous wavelet transforms the choice of the mother wavelet function is only constrained by the admissibility condition. One is free to choose the mother wavelet for optimal behavior in the particular application of interest.

**Mother Mapper Operator**

The Mother Mapper Operator is presented and derived, and its properties are detailed. Several interpretations and applications of this structure are also provided.

The Mother Mapper Operator, \(MM_{g_2}(\cdot)\), maps \((R \setminus \{0\} \times R) \rightarrow (R \setminus \{0\} \times R)\): i.e., it maps a wavelet transform of a function with respect one mother wavelet to a wavelet transform of the same function with respect to a different mother wavelet. The operator is defined as:

\[MM_{g_2} : W_{g_1} f(a, b) \mapsto W_{g_2} f(a, b)\]

\[MM_{g_2} : W_{g_1} f(a, b) \mapsto \frac{1}{c_g} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{ds}{s^2} \frac{W_{g_2} g_2(s, \tau-b)}{a} W_{g_1} f(s, \tau) d\tau\right) d\tau\]

where \(W_{g_2} f(a, b)\) is the wavelet transform of some function, \(f\), with respect to the mother wavelet, \(g (g_2)\) and \(f\) are both functions of the independent variable(s) such as time and/or space). The independent variables are not used in this notation to avoid loss of generality. The positive real constant, \(c_g\), is a normalizing term which is a function of the original mother wavelet only. This constant was defined above as the admissibility constant and was discussed there. The kernel of the Mother Mapper Operator is thus the wavelet transform of the new mother wavelet, \(g_2\), with respect to the old mother wavelet, \(g\). In this operator, the kernel is conjugated and then mapped via an affine transformation over all \((s, \tau)\). Next, these affine mapped kernels are functionally (or term-by-term in the discrete case) multiplied by the original wavelet transform and then integrated. Since the kernel of this new operator is a wavelet transform of the new mother wavelet with respect to the old mother wavelet, it is possible that it will have approximately compact support (or some type of structured or limited support) which may lead to enhanced efficiencies in implementations.

**Mother Mapper Derivation**

The resolution of identity theorem (which is a special case of the Frobenius-Schur-Godement Theorem in square integrable group theory [7]) for the wavelet transform inner products states that when \(g\) is an admissible function then:

\[c_g \langle f, g_2 \rangle = \int \frac{da}{a^2} \int \langle f, g_{a,b} \rangle \langle g_{a,b}, g_2 \rangle db\]

The admissibility constant, \(c_g\), must be a well defined constant (less than infinity). From the definition of the inner product:

\[\langle g_{a,b}, g_2 \rangle = \langle g_2, g_{a,b} \rangle^*\]

so that:

\[\langle f, g_2 \rangle = \frac{1}{c_g} \int \frac{da}{a^2} \int \langle f, g_{a,b} \rangle \langle g_2, g_{a,b} \rangle^* db\]

Under the assumption that \(g_2\) is an admissible function, formulate the new wavelet transform (of \(f\) with respect to the new mother wavelet, \(g_2\)) as an inner product:

\[W_{g_2} f(s, \tau) = \langle f, g_{s,\tau} \rangle\]

where \(s\) is the new scale variable and \(\tau\) is the new translation variable. Now, in the resolution of identity
formulation this creates a term \( W_g g_z(a, b) \) which can be expanded as:

\[
(\sqrt{s} g_z(st+\tau), \sqrt{a} g(at+b))
\]

\[
= \left( g_z(t), \sqrt{\frac{a}{s}} g\left( \frac{a}{s} t + \frac{(b-\tau)}{s} \right) \right)
\]

thus yielding the Mother Mapper Operator:

\[
W_g f(s,t) = \frac{1}{c_g} \int \frac{da}{a^2} (W_g f(a,b)) W_g g_z\left( \frac{a}{s}, \frac{(b-\tau)}{s} \right) db
\]

Thus, the Mother Mapper Operator performs the mapping of the wavelet transform with respect to \( g \) to a new wavelet transform with respect to mother wavelet \( g_z \).

### Multiresolution Orthogonal Wavelets versus Nonorthogonal Wavelets

The Multiresolution orthogonal wavelet transforms presented by Meyer, Mallat, Daubechies and others \([2, 11, 12]\) yield very specific (and sometimes unique) mother wavelets due to the extensive constraints imposed upon them. Figure 1 identifies some of these constraints and the paths taken to create the Multiresolution orthogonal wavelets. This author believes that these are appropriate for signal/image analysis but may be augmented by the nonorthogonal wavelets to create further efficiencies. By using nonorthogonal wavelet transforms the mother wavelets are simply constrained to be admissible functions (bandpass). The freedom in choosing a mother wavelet may be exploited to better match the signal/image characteristics and to more efficiently concentrate the signal's energy into fewer coefficients.

### Mother Mapper Operator Applications

An advantage of using the Mother Mapper Operator to easily and efficiently mapping between transforms with different mother wavelets is that different parameterizations by different mother wavelets may provide efficient representations of the original process. Figures 2a and 1b display wavelet transforms of the same Gaussian windowed tone signal but with respect to different mother wavelets. Figure 2a uses the Morlet, Gaussian weighted tone, and Figure 2b uses a Gaussian weighted linear FM as the mother wavelet, respectively. Figure 2c displays the wavelet transform of a Gaussian weighted linear FM signal with respect to a Gaussian weighted linear FM mother wavelet - thereby creating a wideband cross ambiguity function (WBCAF) of the FM signal \([13]\). These figures emphasize the effects of the mother wavelet choice and how it parameterizes the wavelet transform.

Some processes will be "well represented" by a particular mother wavelet and other processes are poorly represented by that same mother wavelet. The concentration of energy in the transform domain represents the measure of efficiency or performance since a smaller number of coefficients would be required to represent the signal to the same degree of approximation.
The efficiency provided by the Mother Mapper Operator is that it avoids saving the original data or reconstructing that data from a previous wavelet transform. Figure 3 shows the concepts pictorially. Referring to the top row of Figure 3, first take a wavelet transform of the input signal with respect to a initial mother wavelet; if the transform domain representation is "good enough" then stop and use that representation. If the representation is not good enough, then another mother wavelet might be used to decompose the signal (the idea is to proceed to the bottom right corner of Figure 3). At this point, without utilizing the Mother Mapper Operator, either the original data would have had to have been saved or that data would have to be reconstructed from the first wavelet transform. Saving the original data involves large resources as does reconstructing the data from the first transform. Instead, the Mother Mapper Operator provides a more efficient alternative by directly mapping the first wavelet transform to a new wavelet transform with respect to a different mother wavelet kernel. Thus the efficiency of evaluating several transforms with respect to multiple mother wavelets is significantly improved.

The "trick" to efficient wavelet representations (a good approximation of the original signal with few wavelet domain coefficients) is to find the "optimum" mother wavelet for the signals being analyzed (the optimality criteria are arbitrary and chosen by the user’s applications). Obviously, these mother wavelets themselves may be optimized. Specific optimization criteria or algorithms are not addressed and are deferred to future research; however, the implementation of these algorithms would require efficient evaluations of wavelet transforms with respect to different mother wavelets.

Another advantage of changing mother wavelets is that the mother wavelet parameterizes the transform in a manner similar to parametric spectral estimation. If the signals being transformed are similar to the mother wavelet, then the transform domain representation will tend to be efficient and characterize the salient properties of the signal or system. If one portion of a signal can be very efficiently characterized by a first, arbitrary, mother wavelet but another portion of the signal is inefficiently represented by this first mother wavelet, then the overall representation (the mother wavelet and its associated coefficients) will be inefficient. However, if a second mother wavelet is used which can efficiently characterize the originally poorly represented portion of the signal, then two (or multiple) mother wavelets together may be used to yield an extremely efficient representation. See Figure 4 for a diagram of this multiple mother wavelet decomposition process.

The new representations will involve sets containing the significant wavelet coefficients, with one set of coefficients associated to each mother wavelet. Effectively each mother wavelet will extract only that portion of the signal which it can efficiently represent, and it will allow the rest of the signal to be processed by
another different mother wavelet transform. The set of mother wavelets and their associated coefficients (which are hopefully small in number) is the new representation of the signal. Thus a new decomposition, not only in terms of wavelet coefficients but also in terms of mother wavelets, may be defined.

Conclusions

Wavelet theory is extended to include a new operator, the Mother Mapper Operator. This operator allows sets of mother wavelets to be considered rather than just a single mother wavelet. The Mother Mapper Operator can improve the efficiency of implementing several wavelet transforms on the same data set with different mother wavelets (see Figure 4). The Mother Mapper Operator allows a direct mapping from a previously computed wavelet transform to a new transform with respect to a new mother wavelet. The implementation is more efficient since the original space and/or time data does not need to be saved or reconstructed from the original wavelet transform in order to compute the new wavelet transform.

Nonorthogonal wavelet transforms are utilized rather than orthogonal wavelet transforms to provide the freedom in choosing the mother wavelet rather than being highly constrained and yielding a very specific (and possibly unique) mother wavelet.

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References