State-Space Representations of 2-D FIR Lossless Matrices and their Application to the Design of 2-D Subband Coders*

S. Venkataraman and B. C. Levy
Department of Electrical Engineering and Computer Science
University of California, Davis CA 95616

1 Introduction
Digital multirate filter banks are widely used for subband coding of speech signals [2]. FIR lossless systems form a class of perfect reconstruction filter banks, and 2-D filters of this type have been employed in image coding [11]. The advantages of such filter banks include the fact that the analysis and synthesis filters have the same length, which makes their implementation simple, and the stability of filters is guaranteed [9]. Lately there has been a revival of interest in lossless systems. In addition to subband coding, researchers are finding new applications for such systems. For instance, the design of wavelets for the multiresolution analysis of signals or images [6] is closely tied to the design of lossless filter banks.

The properties of 1-D FIR lossless filter banks have been studied extensively, and many efficient procedures have been developed for their design [3, 9]. The design of lossless filter banks often involves constructing lossless transfer matrices whose rows are formed by the polyphase components of each filter in the filter bank. Perhaps, the most important result in the design of such matrices is the ability to express lossless transfer matrices of a certain degree as a cascade of systems with smaller degrees [3]. However, 2-D FIR lossless systems have not been studied as closely as their 1-D counterparts. This paper provides a general characterization of \( p \times p \) lossless 2-D FIR transfer matrices of arbitrary degree in \((z_1, z_2)\). The properties of 2-D state-space realizations of such matrices are examined. It is shown that these matrices are not necessarily decomposable as a product of 1-D matrix transfer functions depending on \( z_1 \) and \( z_2 \) separately. An approach for the design that does not exclude any lossless filter of given degree is presented. An illustrative example which covers a subset of the ideas discussed is presented. This example consists of a two-band lossless filter bank suitable for hexagonal subsampling.

2 First Level Realization
Given a rational 2-D matrix transfer function, a first-level state-space realization is a realization with respect to one of its variables, say \( z_2 \), involving coefficients that are rational functions of the other variable. An important result of 2-D system theory is that it is always possible to obtain a minimal first-level realization of a 2-D rational transfer matrix, but the same cannot be said of second-level realizations, i.e. state-space realizations with respect to both variables. We consider here 2-D causal FIR lossless transfer matrices. It turns out that the first level state-space realizations of such matrices provide substantial information about their structure, which can then be used to develop a design methodology for their synthesis. Another motivation for using a state-space approach is that alternative representations of lossless transfer matrices, such as cascade lattice structures can be derived easily in the state-space framework.

2.1 Characterization of 2-D FIR Lossless Transfer Matrices
Consider an arbitrary \( p \times p \) 2-D FIR quarter plane causal transfer matrix
\[
H(z_1, z_2) = \sum_{j=0}^{k} \sum_{i=0}^{l} h_{ij} z_1^{-i} z_2^{-j},
\]
where \( k \) and \( l \) denote the highest powers of \( H \) in \( z_1^{-1} \) and \( z_2^{-1} \), respectively. Let also \( q \) and \( r \) be the McMillan degrees of \( H \) in \( z_1 \) and \( z_2 \). Then, as shown in [12, 8], it is always possible to construct a minimal first-level state-space realization
\[
H(z_1, z_2) = D(z_1) + C(z_1) (z_2 I_r - A(z_1))^{-1} B(z_1)
\]
(2)
degree \( r \), where \( A(z_1), B(z_1), C(z_1) \) and \( D(z_1) \) are causal rational matrices in \( z_1 \). In the following, such a realization will be denoted as
\[
H(z_1, z_2) \sim M(z_1) = \begin{bmatrix} A(z_1) & B(z_1) \\ C(z_1) & D(z_1) \end{bmatrix}.
\]
(3)
The representation (2) is valid for values of \( z_1 \) that do not correspond to poles of \( M(z_1) \). If the transfer matrix (1) is lossless, following a procedure outlined in [12], it can be shown that there exists a minimal realization \( M(z_1) \) of size \((r+p) \times (r+p)\) which is paraunitary, i.e.,
\[
M(z_1)M^T(z_1^{-1}) = I_{r+p}.
\]
(4)
Of course, the converse is also true, i.e. given a paraunitary matrix \( M(z_1) \), the corresponding matrix transfer
function $H(z_1, z_2)$ is lossless. Furthermore, it can be shown [10] that the causal FIR property of $H(z_1, z_2)$ implies that the above minimal paraunitary realization is causal FIR and of McMillan degree $q$ in $z_1$.

### 2.2 Factorizability

The paraunitarity of $M(z_1)$ implies

$$A(z_1)A^T(z_1^{-1}) + C(z_1)C^T(z_1^{-1}) = I_r. \quad (5)$$

It is also shown below that we can always find a rational paraunitary similarity transformation $T(z_1)$, such that after transformation the matrix $A(z_1)$ is triangular. This transformation preserves the paraunitarity of $M(z_1)$, but its entries become rational functions of $z_1$. From identity (5) and the triangularity of $A(z_1)$, the Theorem 2.4 of [1] implies that the 2-D FIR lossless transfer matrix $H(z_1, z_2)$ can be factored minimally as a cascade of $r$ FIR $p \times p$ lossless sections of degree one in $z_2$, with coefficients which are rational functions of $z_1$.

In order to express the transfer matrix $H(z_1, z_2)$ as a cascade of degree one transfer matrices in one variable, we must first find a similarity transformation $T(z_1)$ such that in the transformed matrix

$$M(z_1) = \begin{bmatrix} \hat{A}(z_1) & \hat{B}(z_1) \\ \hat{C}(z_1) & \hat{D}(z_1) \end{bmatrix} = T(z_1)M(z_1)T^{-1}(z_1) \quad (6)$$

$\hat{A}(z_1)$ is triangular. When $A(z_1)$ is constant, this can be achieved by using the Schur decomposition method. In general, it is not always possible to triangularize a square matrix whose entries are polynomials in $z_1^{-1}$ by using a rational transformation $T(z_1)$. Fortunately, here $H(z_1, z_2)$ is FIR, so that $A(z_1)$ is nilpotent. This implies that all the generalized eigenvectors of $A(z_1)$ are polynomial in $z_1^{-1}$, so that there exists a rational triangularizing transformation $T(z_1)$.

In addition to triangularizing $A(z_1)$, we seek to preserve the paraunitary property of $M(z_1)$ after applying the similarity transformation. According to [12], $M(z_1)$ will remain paraunitary if and only if $T(z_1)$ is itself a paraunitary matrix of size $r \times r$. The fact that the eigenvectors of $A(z_1)$ are polynomial allows the construction of a rational paraunitary $T(z_1)$ which yields a Schur like decomposition of $A(z_1)$. This implies that in the minimal factorization of $H(z_1, z_2)$, the degree one sections in $z_2$ will have rational coefficients in $z_1$.

In summary, given an arbitrary 2-D FIR lossless transfer matrix with minimal paraunitary realization $M(z_1)$, the matrix $A(z_1)$ appearing in such a realization has polynomial eigenvectors. This property implies that $H(z_1, z_2)$ admits a cascade decomposition in degree one sections in $z_2$ whose coefficients are rational functions of $z_1$. If the eigenvectors of $A(z_1)$ are constant, the cascade decomposition involves only FIR, i.e. polynomial in $z_1^{-1}$, coefficients. Furthermore, if $A(z_1)$ itself is constant, and either $C(z_1)$ or $B(z_1)$ is constant, then $H(z_1, z_2)$ is separable and can be expressed as

$$H(z_1, z_2) = H_1(z_1)H_2(z_2). \quad (7)$$

In the subband coding context, the entries of $H(z_1, z_2)$ represent the polyphase components of the analysis filters. Thus, although the matrix $H(z_1, z_2)$ given by (7) is separable, the polyphase components and analysis filters are not separable, independently of the type of subsampling scheme considered.

### 3 Synthesis Based on Roesser's Model

The matrix $A(z_1)$ plays a crucial role in determining the structure of the lossless matrix $H(z_1, z_2)$. This role can be described more explicitly by considering the Roesser state-space model of $H(z_1, z_2)$. Roesser's model provides also a convenient framework for formulating the design of 2-D FIR lossless transfer matrices.

#### 3.1 Roesser's Model for 2-D FIR Lossless Matrices

Roesser's state-space model of a 2-D system [7] is given by

$$\begin{bmatrix} z_1(i+1, j) \\ z_2(i, j+1) \\ y(i, j) \end{bmatrix} = \begin{bmatrix} F_1 & F_2 & G_1 \\ F_3 & F_4 & G_2 \\ H_1 & H_2 & J \end{bmatrix} \begin{bmatrix} z_1(i, j) \\ z_2(i, j) \\ u(i, j) \end{bmatrix}, \quad (8)$$

where $u(i, j)$ and $y(i, j)$ are the input and output vectors, and $z_1(i, j)$ and $z_2(i, j)$ are the horizontal and vertical state vectors. All the matrices are assumed to have compatible sizes. In the following, this model will be denoted as

$$R \triangleq \begin{bmatrix} F_1 & F_2 & G_1 \\ F_3 & F_4 & G_2 \\ H_1 & H_2 & J \end{bmatrix} \triangleq \begin{bmatrix} F & G \\ H & J \end{bmatrix}. \quad (9)$$

In the case of FIR transfer matrices, the size of $F$ indicates whether the realization is minimal or not. If the transfer matrix $H(z_1, z_2)$ has McMillan degrees $q$ and $r$ in $z_1$ and $z_2$, then in a minimal state-space realization, $F$ must have size $(q+r) \times (q+r)$.

The first level realization (2) can be written in terms of Roesser's model (8) as

$$M(z_1) = \begin{bmatrix} A(z_1) & C(z_1) \\ B(z_1) & D(z_1) \end{bmatrix} = \begin{bmatrix} F_4 & G_2 \\ H_2 & J \end{bmatrix} \times \begin{bmatrix} F_1 \end{bmatrix} (z_1I_B - F_1)^{-1} \begin{bmatrix} F_2 & G_1 \end{bmatrix}. \quad (10)$$

As indicated in Section 2.1, we can find a matrix $M(z_1)$ which is causal, paraunitary and of degree $q$. Therefore, there exists a Roesser realization $R$ which is orthonormal and of size $(q+r+p) \times (q+r+p)$.

#### 3.2 Design Approach

To obtain a 2-D FIR lossless transfer matrix, one approach would be to construct a $(p+r) \times (p+r)$ causal FIR lossless transfer matrix $M(z_1)$ of degree $q$ by using
any of the 1-D design techniques available in the literature. However, to ensure that the 2-D transfer matrix is FIR, the $r \times r$ matrix $A(z_1)$ must be nilpotent.

A design technique similar to the 1-D method proposed in [5] relies on Roesser's state-space model. The quantity that we seek to compute is the $(q+p+r) \times (q+p)$ matrix $\mathcal{R}$ given by (10). Losslessness is ensured by making $\mathcal{R}$ orthonormal. Then, if we consider

$$F = \begin{bmatrix} F_1 & F_2 & F_3 \\ F_3 & F_4 & F_5 \\ F_5 & F_6 & F_7 \end{bmatrix},$$

(11)

a set of necessary and sufficient conditions for $H$ to be FIR is that

- $A(z_1) = F_1 + F_3(z_1 I_q - F_1^{-1}) F_2$ is $r \times r$ and nilpotent of order $r$.
- $A(z_1)$ is made nilpotent of the required order by setting the traces of $A^n$ for $n = 1, 2, \ldots, r$ equal to zero. These conditions guarantee that in our search for optimal filters, no $p \times p$ 2-D FIR lossless matrix is excluded.

Up to this point, we have considered general 2-D lossless FIR transfer matrices which involved constraints which are not simple to enforce. Let us consider now a subclass of such matrices, associated to a special structure of the matrix $F$ in (11). This subclass corresponds to the case discussed in Section 2.2, where the eigenvectors of the matrix $A(z_1)$ are constant.

Since the matrices $F_1$ and $F_4$ in (11) can be assumed strictly lower triangular, the condition that $H(z_1, z_2)$ should be FIR can be expressed as

$$\det \left( \left( z_2 I_q - F_4 \right) - (F_3(z_1 I_q - F_1^{-1}) F_2) \right) = z_2^r.$$  

(12)

In this expression, the matrices $z_2 I_q - F_4$ and $(z_1 I_q - F_1^{-1})$ are invertible and lower triangular. Then, a sufficient condition for (12) to be satisfied is that $F_2$ and $F_3$ should be strictly lower triangular. In fact, this requirement can be loosened slightly by observing that one of the two can have several nonzero superdiagonals, if the other one is strictly lower triangular with an equal number of zero subdiagonals. This yields a matrix $F(z_1)$ which is lower triangular. Then, it can be shown [10] that 2-D FIR lossless transfer matrices which satisfy the above conditions and $F_1, F_2, F_3$ and $F_4$ can be factored as a cascade of lossless sections of degree one in $z_1$, whose coefficients are polynomial in $z_1^{-1}$.

### 3.3 Separable FIR Lossless Transfer Matrices

The class of 2-D lossless transfer matrices considered by Karlsson and Vetterli [4] corresponds to a further subclass of the one we have just considered, where $F_1$ and $F_4$ are strictly lower triangular, $F_2 = 0$ and $F_3$ is arbitrary. From expression (10) for the first-level realization, we see that in this case $A(z_1) = F_4$ is a constant matrix. Consider the factorization

$$\begin{bmatrix} F_3 & G_2 \\ H_1 & I \end{bmatrix} = \begin{bmatrix} L_2 & J_2 \\ L_1 & J_1 \end{bmatrix},$$

(13)

where the matrices appearing on the right hand side of (13) have dimensions $(q + p) \times t$ and $t \times (p + r)$, respectively. Substituting $F_2 = 0$ and (13) inside the formula for the transfer matrix

$$H(z_1, z_2) = J + \begin{bmatrix} H_1 & H_2 \end{bmatrix} \times \left( \begin{bmatrix} z_2 I_q & 0 \\ 0 & z_2 I_r \end{bmatrix} - F \right)^{-1} \begin{bmatrix} G_2 \\ G_1 \end{bmatrix},$$

(14)

yields

$$H(z_1, z_2) = \begin{bmatrix} J_2 + H_2(z_2 I_q - F_4)^{-1} L_2 \\ J_1 + L_1 (z_1 I_q - F_1)^{-1} G_1 \end{bmatrix} \triangleq H_2(z_2) H_1(z_1).$$

(15)

Thus the $p \times p$ transfer matrix $H(z_1, z_2)$ can be decomposed as a product of two 1-D transfer matrices $H_1(z_1)$ and $H_2(z_2)$ of sizes $p \times t$ and $t \times p$, respectively, where $t$ is the rank of the matrix appearing on the right hand side of (13).

However, in constructing the factorization (13), we have not used the fact that the 2-D FIR transfer function $H(z_1, z_2)$ is lossless, or equivalently that the Roesser realization $\mathcal{R}$ is orthonormal. When this property is taken into account, it can be shown [10] that if $t = p$, so that in (15), the 2-D transfer matrix $H(z_1, z_2)$ has been factored as the cascade of two 1-D $p \times p$ FIR lossless transfer matrices $H_1(z_1)$ and $H_2(z_2)$ of degrees $q$ and $r$ in $z_1$ and $z_2$, respectively.

Remark: The number degrees of freedom available for this structure is given by

$$N_s = \frac{1}{2}[(p+q+r)(p+q+r-1)-(q+r)(q+r+1)].$$

All the structures in which unit degree factors in $z_1$ and $z_2$ are cascaded have the same degrees of freedom for a given $q$ and $r$. But the order in which they are cascaded must be construed as degrees of freedom.

### 4 Design Examples

We design a 2-D FIR lossless filter bank $h(z_1, z_2) = \begin{bmatrix} h_0(z_1, z_2) \\ h_1(z_1, z_2) \end{bmatrix}$ for subband coding of rectangularly sampled 2-D input signals. This filter bank is suitable for hexagonal (quincunx) subsampling by a factor two using the sampling matrix $D_s = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. The polyphase vectors associated with $D_s$ are $\{1\}$ and $\{z_1^{-1}\}$. $h(z_1, z_2)$ can be expressed in terms of its polyphase components and polyphase vectors as

$$\begin{bmatrix} h_0(z_1, z_2) \\ h_1(z_1, z_2) \end{bmatrix} = H(z_1 z_2^{-1}, z_1 z_2) \begin{bmatrix} 1 \\ z_1^{-1} \end{bmatrix},$$

(17)

and

$$H(z_1, z_2) \triangleq \begin{bmatrix} h_{00}(z_1, z_2) & h_{01}(z_1, z_2) \\ h_{10}(z_1, z_2) & h_{11}(z_1, z_2) \end{bmatrix}. $$

(18)
A FIR lossless $H(z_1, z_2)$ was designed, which upon substitution in (17) yields a perfect reconstruction filter bank. 2-D band splitting is accomplished by requiring the passband of $h_0(z_1, z_2)$ to be restricted to the region enclosed by the square connecting the four points $(-\pi, 0), (0, \pi), (\pi, 0)$ and $(0, -\pi)$ in the 2-D Fourier plane. Three different matrix filters $H(z_1, z_2)$ with $q = r = 2$, $q = r = 4$, and $q = r = 8$ were designed. The low- and high-pass filters $h_0(z_1, z_2)$ and $h_1(z_1, z_2)$ corresponding to these three matrix filters have roughly 8, 32 and 64 coefficients, respectively. The state-space methods described in the previous section were employed for the design of a matrix $R$, where depending on whether $q = r = 2$, $q = r = 4$ or $q = r = 8$, $R$ has dimension $6 \times 6$, $10 \times 10$, and $18 \times 18$, respectively. The frequency sampling filter design method was employed, and optimization was carried out by using the MINOS nonlinear optimization software package. The 2-D magnitude responses of $h_0(z_1, z_2)$ for each of the three designs are plotted in Figs. 2, 4, and 6. A cross section of the magnitude response plot along the $\omega_1 = \omega_2$ axis is shown in Figs. 1, 3, and 5, which displays precisely the filter attenuation in the stop band. It can be seen from these figures that the ideal low- and high-pass filter characteristics are approximated increasingly accurately as the filter degrees $q$ and $r$ are increased. The wing-like behavior appearing in the 2-D magnitude response plot is due to the fact that the transition band between the pass and stop bands was kept very narrow.

5 Conclusions

The properties of first-level realizations of 2-D FIR lossless transfer matrices were examined. These properties were used to characterize the structure of 2-D Roesser state-space models, and to develop a general design methodology for such transfer matrices. The results described here are directly applicable to the design of 2-D quadrature mirror filters. They can also be employed to generate 2-D nonseparable orthogonal wavelets. For instance, with $\omega = [\omega_1, \omega_2]^T$, the inverse Fourier transform of

$$\\phi(\omega) = \prod_{k=1}^{\infty} h_0((D_z)^{-T})^k\omega, \quad (19)$$

gives a possible choice [5] for the generating function $\phi(x_1, x_2)$, which along with its translates and dilates, spans the space $L^2(\mathbb{R}^2)$. In the above expression, $h_0(z_1, z_2)$ is the lowpass filter of a 2-channel lossless filter bank. Note that regularity constraints need also to be imposed on $h_0$ to ensure that the function $\phi(x_1, x_2)$ is smooth. The corresponding wavelet $\psi(x_1, x_2)$ generating the wavelet bases can then be expressed in terms of $\phi(x_1, x_2)$.

A number of issues remain to be investigated, such as the relation between 2-D state space models and lattice representations of 2-D lossless transfer matrices. The design methodology presented in this paper requires the use of constrained optimization techniques to exploit all the available degrees of freedom. Among the design constraints listed in Section 3.2, the orthonormality property of the matrix $R$ can easily be imposed structurally by expressing it as the product of planar rotation matrices [9]. On the other hand, it is unclear whether the trace conditions on the matrix $A(z_1)$ can be expressed structurally, thus allowing the use of unconstrained optimization techniques.

References


Figure 1: Magnitude response cross section along the $\omega_1 = \omega_2$ axis for the case $q = r = 2$. The middle plot represents the low-pass filter.

Figure 2: Magnitude response of $h_0(z_1, z_2)$ for the case $q = r = 2$.

Figure 3: Magnitude response cross section along the $\omega_1 = \omega_2$ axis for the case $q = r = 4$. The middle plot represents the low-pass filter.

Figure 4: Magnitude response of $h_0(z_1, z_2)$ for the case $q = r = 4$.

Figure 5: Magnitude response cross section along the $\omega_1 = \omega_2$ axis for the case $q = r = 4$. The middle plot represents the low pass filter.

Figure 6: Magnitude response of $h_0(z_1, z_2)$ for the case $q = r = 8$. 