Multiresolution Transform Matrices

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Abstract

Matrix descriptions of subband coders are introduced and used to obtain transform matrices of pyramidal and related algorithms. These matrices are generalizations of the Haar matrix, and they are all shown to have a stratified structure related to circulant matrix structure. An implementation of a class of biorthogonal digital filter banks (DFB's) with an efficient predictive structure is introduced. DFB's and pyramids optimal for quantization are also introduced.

Introduction

Multiresolution methods for image processing have been under development since the 1970's, but these methods date back at least to early work of Haar in the 1920's. The prototype multiresolution transform matrix is the Haar matrix [1] shown in (1) with spacing to emphasize its stratification. Its corresponding fast algorithm is now referred to as a pyramidal algorithm (or simply a pyramid) [2]. Recent research in the area of wavelets [3-8] has identified new orthogonal and biorthogonal multiresolution transforms, generalizing the Haar transform.

In this paper transform matrices of multiresolution transforms are introduced and illustrated with a new pyramidal algorithm with reduced computational load. This algorithm is based upon optimal linear estimation methods and coset predictors, recently introduced [9].

Matrix Descriptions

Coders Based On Biorthogonal Pairs of Matrices

In this and succeeding sections matrices will be introduced which describe multi-resolution transforms of signal vectors. This is illustrated in Figure 1a, in which x, u, v, and y, are all assumed to be row vectors of length N of signal components, e.g. \( x = [x_0, x_1, \ldots, x_{N-1}] \). If \( v = u \), it will be assumed that the output, \( y \), equals \( x \), and, since they are row vectors, \( y = xA^*B \), and therefore \( A^*B = I_n \), the unit matrix of order N. This describes a perfect reconstruction (PR) coder. An adjoint coder, shown in Figure 1b, is obtained by taking the Hermitian transpose giving

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\end{bmatrix}
\]
B*A = I, A pair of matrices, A and B, satisfying A* = BJ are said to be bi-orthogonal. In the unitary case, A = B, and the two coders coincide.

A = B*  u  v  B  y

Channel (a)

B = A*  u  v  A  y

Channel (b)

Figure 1. A Coder (a) and its Adjoint (b) with a Pair of Biorthogonal Matrices

Digital Filter Banks

A matrix description of a two-channel analysis/synthesis digital filter bank (DFB) is shown in Figure 2(a). The vectors u and v are of length M = N/2, and therefore, the H_i and G_i are M x N rate increasing matrices and their Hermitian conjugates, H_i* and G_i*, rate decreasing. For this DFB to be perfect reconstruction,

\[ [H_0^*, H_1^*] \begin{bmatrix} G_0 \\ G_1 \end{bmatrix} = H^*G = I_N, \]

which defines H and G. Equivalently,

\[ H_0^*G_0 + H_1^*G_1 = P_{10} + P_{01} = I_N, \]  (3)

which defines the two matrices, P_{ij}. These necessarily satisfy P_{ij}P_{ij} = P_{ij}^T, i.e., they are idempotent matrices of projectors [10]. Since a matrix necessarily commutes with its inverse

\[ H^*G = GH^* = I_N, \]  (4)

giving four conditions equivalent to (3),

\[ G_iH_i^* = \delta_{ij}I_M, \]  (5)

for i,j = 0,1 and \( \delta_{ij} \) the Kronecker delta. From \( J = I \), it is seen that \( G_i \) is a left inverse of \( H_i^* \), for \( i = 1,2 \). Equation (5) states necessary and sufficient conditions for the structure in Figure 2a to provide perfect reconstruction.

From the preceding discussion, it is evident that the associated synthesis/analysis DFB, shown in Figure 2b, is also PR. In addition to these two DFB's are their adjoints obtained by interchanging the H_i and G_i. It is evident that all these fit the description of Figure 1 if the definitions \( u = [u_0, u_1] \) and \( v = [v_0, v_1] \) are adopted in Figure 2a with similar definitions for x and y in Figure 2b.

\[ \begin{bmatrix} H_0^* & u_0 \\ v_0 & G_0 \end{bmatrix} y^0 \]

Analysis (a)

\[ \begin{bmatrix} H_1^* & u_1 \\ v_1 & G_1 \end{bmatrix} y^1 \]

Synthesis (b)

Figure 2. Analysis/Synthesis and Synthesis/Analysis DFB's Resulting from Equation (4).

A general expression for a left inverse will prove useful in identifying the \( G_i \) which are left inverses of \( H_i^* \),

\[ G_i = K_iH_i^*, \]  (6)

in which

\[ K_i = H_iJ_iH_i^*, \]  (7)

for any \( J_i \) such that \( K_i \) is invertible. It is evident that \( G_iH_i^* = I_{M'} \), however, the converse is easily shown by assuming \( J_i = G_i^*G_i \) if \( G_i \) is known, indicating that any left inverse can be expressed as in (6) [11].

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It can be seen that given \( H \) of full rank that any complementary matrix, \( H' \), may be chosen such that \( H \) is invertible. Equation 2 then uniquely defines \( G \) and hence \( G_o \) and \( G_r \). There are therefore many matrices satisfying (2) - (5). Further properties of these matrices, appropriate to their roles in DFB's and wavelet transforms, are to be discussed in the next section, and these give guidance in their determination.

**Circulant and Rate-Changing Matrices**

**Permutation Matrices**

The matrices and projectors just described have been under investigation in a related problem, that of designing error-correcting codes that work over the real number field rather than over finite fields [12-14]. In this application as well as the application at hand, two types of matrices appear, incidence matrices, denoted \( M_i \), and Toeplitz matrices associated with a z-transform, \( H(z) \), denoted \( H(S) \). In the finite block length case for one-dimensional systems, \( S \) is the circular shift matrix

\[
S = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(8)

\( H(S) \) is then a polynomial in powers of \( S \), a circulant matrix [15]. \( S \) is a permutation matrix, each of the \( N \) rows being a distinct length \( N \) unit vector, denoted \( e_r \). A second permutation matrix for the one-dimensional case, \( M \), consists of two \( M \times N \) incidence matrices, \( M_i \),

\[
M = \begin{bmatrix}
M_0 \\
M_1
\end{bmatrix}
\]

(9)
in which

\[
M_0^T = [e_0^T, e_2^T, \ldots, e_{N-2}^T]
\]

(10)

and \( M_1 = M_0 S^{-1} \) consists of the remaining unit vectors. The multidimensional case was discussed in Reference [9], and examples of the \( M_1 \) were given for quincunx sampling lattices.

**z-Transform Specification of Matrices**

The DFB matrices will be determined and shown to yield the z-transforms in the diagram of Figure 3, in which \( q = 1 - p, p = 0 \) or 1. A delay and advance in one of the paths is explicitly shown, as in [16]. The arrows indicate down and up sampling by a factor of 2, in the even locations. The integers \( p \) and \( q \) can actually be chosen to be any pair of positive or negative integers of opposite parity, without affecting the perfect reconstruction property, a flexibility that can be used to obtain causal solutions (if, a constant delay is also accepted instead of the perfect reconstruction property, assumed here).

To begin, assume

\[
H_0^* = H_0(S)M_p^*,
\]

(11)
in which the circulant matrix, \( H_0(S) \), describes the filtering and the \( M_p^* \) the downsampling with samples in even or odd locations preserved as \( p \) is even or odd. The left inverse of \( H_0^* \), \( G_0 \), will be determined from (6). In order to avoid recursive filtering, it will be required that
Define

\[ K_0 = M_p H_0(S^*) J_0(S) H_0(S) M_p^* = I_m \quad (12) \]

assume \( J_0(z) = J_0(z^r) \), and observe that (12) implies the half-band condition, well-known in perfect reconstruction DFB design [16],

\[ N_0(z) + N_0(-z) = 1 \quad (14) \]

The presence of the factor \( J_0(z) \) ensures that the following development applies to general biorthogonal DFB’s [6-7]. To summarize, the condition \( K_0 = M_p N_0 M_p^* = I_m \) is equivalent to \( N_0(z) \) being half-band.

In a similar manner, \( H_1(z) \) and \( G_1(z) \) are related by

\[ N_1(z) = H_1(z) H_1(z^r) \quad (15) \]

which is also half-band. Comparing (14) with (3) implies that

\[ N_1(z) = N_0(-z) \quad (16) \]

Equation (5) with \( i \neq j \) indicates that these will be satisfied if

\[ G_1(z) = H_0(z^r) \quad (17) \]

since \( G_1 H_1^* \) becomes \( M_q G_1(S) H_1(S) M_q^* \) which is \( 0_m \) since \( G_1(S) H_1(S) \) consists of only even powers of \( S \), and \( M_q S^k M_q^* = 0_m \) for any even \( k \). Lastly,

\[ H_1(z) = H_0(z^r) J_0(z) \quad (18) \]

follows from (15) - (17).

The matrices for a biorthogonal, perfect reconstruction DFB are then of the form:

\[ H_0 = H_0(S) M_p \quad (19a) \]

\[ H_1 = H_1(S) M_q = H_0(S^r) J_0(S) M_q^* \quad (19b) \]

\[ G_0 = M_p G_0(S) = M_p H_0(S^r) J_0(S) \quad (19c) \]

If \( J_0(S) = I_{2n} \) the DFB is orthogonal, as in [16]. One can specify \( H_1^* \) first, instead of (11), if it is desired that \( H_1(z) \) be of lower order than \( H_0(z) \), effectively swapping the \( J_1(z) \) factor between \( H_0^* \) and \( G_0 \) and the \( J_0(z) \) factor between \( H_0^* \) and \( G_1^* \).

The definitions of the z-transforms in Figure 3 are apparent from (19), and they follow because (19) holds for \( S \) of any size which then specifies the z-transforms arbitrarily densely on the unit circle, and hence over the whole z-plane except points of singularity, the only finite possibility being \( z = 0 \). The equivalence of (12) and (14) also depends upon this argument.

**Multiresolution Structures and Matrices**

**Pyramid Structures**

Multiresolution DFB’s can be formed by repeatedly introducing A/S DFB’s that transform both \( u_i \) and \( u_j \) in Figure 2a, giving tree structures, or that transform only \( u_i \) giving octave band structures [17]. The wavelet transform has been shown to be equivalent to the latter case [4], and, for demonstrating convergence to signals with arbitrarily high bandwidth, is considered in the limit for an infinite number of resolution levels. In the applications, only a finite number of levels need be employed, which permits a matrix description to be introduced that generalizes the Haar matrix.

Following the notation of the wavelet literature, the resolution is identified by an integer, \( m \), and \( 2^m \) indicates the resolution. In our notation, with the original \( N_0 \), the length of \( x \) and \( y \) in Figure 1, denoted \( N(0) \),

\[ N(0) = 2^m N(0) \quad (20) \]

Figure 4 illustrates a system with \( L \) resolution levels, indexed from 0 to \( L - 1 \), provided by concatenating \( L - 1 \) DFB’s in a pyramidal construction.
The DFB at level $L - 2$ is conventional, but at higher resolution levels, only the lower, high-frequency path is present. 

**Stratified g-circulant Matrices**

The analysis output and synthesis input is seen to be of the form

$$v = [v_0^{(L-1)}, v_1^{(L-1)}, v_1^{(L-2)}, \ldots, v_1^{(2)}, v_1^{(1)}]$$

Each input $v_n^{(m)}$ can be seen to contribute to the output $y$ via a vector $q_n^{(m)}$, of length $N^{(m)}$, i.e., $y = \sum_m q_n^{(m)}$. Tracing the typical path gives

$$q_n^{(m)} = v_n^{(m)} B_n^{(m)}$$

$$= v_n^{(m)} [G_1^{(m)} G_0^{(m-1)} \ldots G_0^{(2)} G_0^{(1)}]$$

which defines the $N^{(m)} \times N^{(0)}$ matrix $B_n^{(m)}$. The matrix $B$, which maps $u$ to $y$ therefore has the following stratified structure (shown transposed), e.g., the Haar matrix in Eq. 1:

$$B^T = [B_0^{(L-1)^T} B_1^{(L-1)^T} B_1^{(L-2)^T} \ldots B_1^{(2)^T} B_1^{(1)^T}]$$

$B$ bears an interesting relationship to a g-circulant matrix which is a matrix in which each row is a cyclic right shift of the row above it by $g$, $g = 2^m$ in this case [14]. Assuming $g$ divides $N$, an $N \times N$ g-circulant has $N/g$ identical strata of this form, and $B$ has only one stratum from each of the $L$ g-circulants. The Haar matrix in Figure 1 also illustrates this. Some properties will be stated.

**Properties:**

1. The synthesis matrix $B$ for the pyramid in Figure 4 having $L$ resolution levels is a stratified g-circulant matrix with $L$ strata, and $g = 2^m$ for levels $0 < m < L$ and $g = 2^{L-1}$ for level $L$.

2. The matrix $B = B_1 B_2 B_3 \ldots B_{L-2} B_{L-1}$, in which $B_n = \text{diag}(G_n^{(m)}, I^{(m-1)})$, $n=0, \ldots, L-2$, with $G_n^{(m)}$ being defined, as in (22), of dimension $N^{(m)}$. This description, like Figure 2, defines the fast pyramidal algorithm for the synthesis branch.

3. The matrix $B$ will be orthogonal if each of the DFB’s composing it are orthogonal.

**Optimal Biorthogonal DFB Implementation with Coset Predictors**

The problem of making a minimum-variance estimate of $y$ in Fig. 2a with $y_0$ was considered in [9] for the special case of $H_0 = M_0$, and an efficient coset predictor structure was introduced for implementation. It is significant that the variance of $u_n^{(m)}$ is also minimized, as this facilitates the quantization of $u$. The following properties generalize the former results:

**Properties:**

4. For a given $H_0$, $y^0$ will be a minimum-variance estimate of $y$ if $J_0 = R_{xx}^{\ast}$, the correlation matrix of $x$ (or $J_0(z) = R_{xx}^{\ast}(z)$).

5. If a biorthogonal DFB is defined by (19), if $N_0(z)$ and $H_0(z)H_0(z^\ast)$ are both half-band, and if $C_0 = G_0 G_0^{\ast}$, then the DFB is implemented by the structure in Figure 5.
If \( f(z) = R_x(z) \) is a good model for the signal, \( x \), then the DFB designed with these matrices in (19) should be optimal in the pyramidal structure. \( R_x \) could be chosen differently at each resolution level. An efficient realization of the DFBs in the pyramid using coset predictors has the structure shown in Figure 5.

![Figure 5. Biorthogonal DFB Implemented with Low-Rate Coset Predictors.](image)

References


