Searching for Convergence Points of the Continuous Time Extended Kalman Filter Used as a Parameter Estimator

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Abstract
This paper deals with estimation of two stable pole parameters for a two-dimensional continuous-time linear stochastic system with known process noise covariance, using the Extended Kalman Filter. Averaging theory permits algebraic computation of a vector field whose stable stationary points are the estimator’s only possible convergence points. Specialized partitioned matrix computations allowed the numerical computation of the vector field and graphical computer search for “spurious” convergence points (not corresponding to the true parameter values), with negative results. This supports the conjecture that none exist, a result known from theory in the one-dimensional case.

1 Continuous Time Averaging Theory

Averaging theory for discrete time parameter estimators has been developed by Ljung [1] and applied to the extended Kalman filter used as a parameter estimator (EKFPE) by Ljung and Söderström [2]. Recently there has been a rigorous derivation of averaging theory for continuous time parameter estimators by DeWolf and Wiberg [3].

To introduce the basic idea of averaging as applied to a continuous time process, let $w(t)$ be a zero mean rapidly fluctuating noise signal, and $x(t)$ an $n$-vector satisfying

\[ dx/dt = t^{-1} f(x, w) \]  

Assume that $w(t)$ is stationary with probability density $p(w)$ which does not depend on the state $x$. The $t^{-1}$ factor will force $x(t)$ to tend toward a constant value when $t$ is large. Therefore $x(t)$ is called the slow variable, and $w$ the fast variable. A simple exponential change of the time scale

\[ x_a(t) = x(e^t) \]  

converts (1) to

\[ dx_a/dt = f(x_a(t), w(e^t)) \]  

The rate of fluctuation (i.e. bandwidth) of $w(e^t)$ increases with time. In the limit the noise $w$ averages the function $f$ and asymptotically $x_a(\cdot)$ follows the trajectories of $\bar{x}(\cdot)$ given by

\[ d\bar{x}/dt = \bar{f}(\bar{x}) \]  

where $\bar{f}$ is defined by

\[ f(z) = E_w f(x, w) = \int f(x, w) p(dw) \]  

Thus the asymptotic properties of the solution $x$ of the stochastic equation (1) are characterized by the ordinary differential equation (4).

Most on-line parameter estimators are not in the form of equation (1), but are slightly more complex. Let $\theta$ be the parameter estimate and other visibly slow variables, let $z_1$ be the states, state estimates, and other variables satisfying linear stochastic differential equations, let $z_2$ be the variables of the Riccati and other nonlinear ordinary differential equations, and let $u$ be a deterministic input. Then $\theta$ will be the slow variable, $z = (z_1^T, z_2^T)^T$ will be the fast variable, and the on-line parameter estimator can (usually) be put in the form

\[ d\theta = t^{-1} [ F_1(\theta, z) dt + F_2(\theta, z) dw ] + t^{-2} [ G_1(\theta, z, t) dt + G_2(\theta, z, t) dw ] \]  

\[ dz_1 = A_1(\theta, z_2) dt + B_1(\theta, z_2) dw + t^{-1} [ S_1(\theta, z_2) dt + S_2(\theta, z_2) dw ] \]  

\[ dz_2 = A_2(\theta, z_2) dt + t^{-1} [ H_1(\theta, z_2) dt + H_2(\theta, z_2) dw ] \]  

in which the $A$'s, $B_1$, the $F$'s, $G$'s, $H$'s, and $S$ are locally bounded and have bounded second derivatives,
and furthermore, $S, F_3, F_4, G_1, G_2, H_1$, and $H_2$ are bounded, $F_1, F_2, A_1, A_2,$ and $B_2$ are limited in growth, and $A_1$ is asymptotically stable as described in [3].

Then, if equation (8) is well behaved, as stated in [3], and equation (6) has no finite escape, the ordinary differential equation corresponding to (4) for the system (6-8) is

$$d\theta/dt = \dot{F}(\theta)$$

where $\dot{F}$ is defined by

$$\dot{F}(\theta) = \int F_1(\theta, z) P^\theta(dz)$$

with $P^\theta(\cdot)$ the unique invariant measure of equations (7) and (8) for fixed $\theta$ and $G_1, G_2, H_1, H_2$ set to zero.

The solutions $\theta(t)$ of (6) asymptotically approach the averaged trajectories $\dot{\theta}(t)$ that obey (9), with probability one. In particular, the convergence points of the algorithm (6-8) are the stable stationary points of the ODE (9), given by those solutions $\theta_\infty$ of

$$\dot{F}(\theta_\infty) = 0$$

for which all the eigenvalues of $\partial\dot{F}/\partial\theta(\theta_\infty)$ are in the left half plane.

2 Application to the EKFPE

The above analysis has been applied to the EKFPE used to estimate system parameters including the state noise covariance. In [3] it is concluded that incorrect parameter estimates will result almost always when the state noise covariance is unknown. When the state noise covariance is known, a first order system must have a convergence point at only the true parameter value for the EKFPE [4].

Here algebraic equations are developed for finding the convergence points of an order $n$ EKFPE. For simplicity, this development is for a zero-input EKFPE in canonical form, but it can easily be extended to the general case. Consider the system

$$dx = A(\theta)xdt + \Sigma dw$$

$$dy = e_1^T x dt + \rho dw \quad (\rho > 0)$$

with $\nu(\cdot)$ and $\nu^2(\cdot)$ uncorrelated standard Wiener processes and

$$A(\theta) = A_0 + e_n(\theta)^T$$

where $A_0$ is the zero matrix except for ones on the super diagonal. For the system (12), (13), the EKFPE to estimate $\theta$ is

$$d\dot{x} = [A(\theta)\dot{x} + e_n \text{Tr}(P_{x2})]dt + \rho^{-2}P_{x2}e_1(dy - e^T \dot{x} dt)$$

$$d\dot{\theta} = \rho^{-2}P_{x2}e_1(dy - e^T \dot{x} dt)$$

$$A(\dot{\theta}) = A(\theta) - \rho^{-2}P_{x2}e_1e_1^T$$

$$dP_{z\theta}/dt = A(\theta)P_{z\theta} + e_n e^T P_{z\theta}$$

$$dP_{x\theta}/dt = -\rho^{-2}P_{x\theta}e_n e^T P_{x\theta}$$

These EKFPE equations (15-20) can be put in the form of (6-8) by the change of variables

$$W = t^{-1} \rho^{-1}$$

$$A_{x\theta} = P_{x\theta}P_{x\theta}^{-1} = tP_{x\theta}V$$

With $\dot{x} = x - \dot{x}$ and $d_3 = e_1 e^T (6)$ becomes

$$d \begin{pmatrix} \dot{\theta} \\ V \end{pmatrix} = \frac{1}{\rho^2} \begin{pmatrix} V^{-1}W^T e_1(e_1^T \dot{x} dt + \rho dw) \\ W^T d_3 W - \rho^2 V \end{pmatrix}$$

The fast variables $z_1$ and $z_2$ can be identified as

$$z_1 = \begin{pmatrix} x \\ \dot{x} \\ W e_1 \\ \vdots \\ W e_n \end{pmatrix} \quad \text{and} \quad z_2 = P_{x\theta}$$

Let $\theta_0$ be the true value of $\theta$. For any $n \times n$ matrix $M$, define $\Phi(\theta, M)$ as

$$\begin{pmatrix} A(\theta_0) & 0 & 0 & 0 \\ e_n(\theta_0 - \theta)^T & M & 0 & 0 \\ e_n e_1^T & -e_n e_1^T & M & 0 \\ e_n e_2^T & -e_n e_2^T & 0 & M \\ \vdots & \vdots & \vdots & \vdots \\ e_n e_n^T & -e_n e_n^T & 0 & 0 \end{pmatrix}$$
Then (7) becomes
\[ dz_1 = \Phi(\hat{\theta}, A(\theta))z_1 + \begin{pmatrix} \Sigma & 0 \\ 0 & -\rho^{-1}P_x \end{pmatrix} \begin{pmatrix} dv \\ dw \end{pmatrix} \]
and (8) becomes the matrix differential equation
\[ \frac{dP_x}{dt} = A(\theta)P_x + P_xA^T(\theta) + \Sigma \Sigma^T + \rho^{-2}P_xd_1d_1^T P_x + \tau^{-1}[\Sigma W \dot{z}_n \Sigma + \epsilon_n \dot{z}^T W \Sigma \dot{z}^T] \]
(24)
Restriction of the estimates \( \hat{\theta} \) to a region of \( \theta \)-space for which the EKFPE is asymptotically stable permits the application of averaging theory as described in the previous section [3]. The ODE (9) then becomes
\[ \frac{d\hat{\theta}}{dt} = \rho^{-2} \begin{pmatrix} \hat{V}^{-1}E\{W^T d_1 \dot{z}\} \\ E\{W^T d_1 \dot{W}\} - \rho^2 \hat{V} \end{pmatrix} \]  
(26)
The expectations \( E \) in the above ODE are computed with respect to the steady state variance equations of the fast system (24) and (25). The stationary points of \( \hat{\theta} \) are given by \( \frac{d\hat{\theta}}{dt} = 0 \), and hence, from the first row of (26), they satisfy
\[ 0 = E\{W^T d_1 \dot{z}\} = \begin{pmatrix} E\{W_1 \dot{z}_1\} \\ \vdots \\ E\{W_n \dot{z}_1\} \end{pmatrix} \]  
(27)
To compute the \( E\{W_1 \dot{z}_1\} \) for \( i = 1, \ldots, n \), define the normalized steady state solution \( P_\infty(\theta) \) of (25) as
\[ P_\infty(\theta) = \rho^{-2} \lim_{t \to \infty} P_x(t; \theta) \]  
(28)
and define \( Q \) as
\[ Q = \rho^{-2} \Sigma \Sigma^T \]  
(29)
Then from (17) and the steady state of (25), \( P_\infty = P_\infty(\theta) \) is determined by the algebraic Riccati equation
\[ 0 = A(\theta)P_\infty + P_\infty A^T(\theta) + Q - P_\infty d_1d_1^T P_\infty \]  
(30)
Put
\[ A_\infty(\theta) = A(\theta) - P_\infty(\theta)d_1^T \]  
(31)
and
\[ \Lambda(\theta) = \Phi(\theta, A_\infty(\theta)) \]  
(32)
and
\[ \Gamma(\theta) = \rho^2 \begin{pmatrix} Q & Q \\ Q & Q + P_\infty d_1d_1^T P_\infty \end{pmatrix} \]  
(33)
Let \( \Omega(\theta) \) be the covariance of \( z_1 \)
\[ \Omega(\theta) = E\{(z_1 - E\{z_1\})(z_1 - E\{z_1\})^T\} \]  
(34)
Then the steady state variance equation corresponding to (24) is
\[ 0 = \Omega(\theta) \Lambda^T(\theta) + \Lambda(\theta) \Omega(\theta) + \Gamma(\theta) \]  
(35)
Since the stationary point condition (27) can be written as
\[ \rho^{-2}(0 0 I 0 \ldots 0) \Omega(\theta) \begin{pmatrix} 0 \\ \epsilon_1 \\ \vdots \\ 0 \end{pmatrix} = 0_n \]  
(36)
the above two equations determine the convergence points of the EKFPE (15-20) and depend only on \( \Sigma \) and \( \theta_0 \) and not on \( \rho \).

3 Computing EKFPE Convergence Points

Direct computation of the convergence points \( \theta_\infty \) involves solution of the algebraic Riccati equation (30) for \( P_\infty(\theta) \). This is analytically intractable for \( n \geq 2 \). Consequently a numerical attack on the problem is required.

In outline, the numerical method chosen is to evaluate (35) for values of \( \theta_i \) in a grid, and then determine the extent to which the stationary point condition (36) is satisfied. In practice, only a portion of \( \Omega \) needs to be computed in order to determine the \( n \)-vector \( L_i \) which is the left side of equation (36). The flow of computation can be listed as:

For grid points \( i = 1, 2, \ldots, N \),
1. Select \( \Sigma, \theta_0, \) and \( \theta_i \).
2. Compute \( A(\theta_i) \) and \( A(\theta_0) \) from (14).
3. Solve for \( P_\infty(\theta_i) \) from (30).
4. Compute \( A_\infty(\theta_i) \) from (31).
5. Compute \( \Lambda(\theta_i) \) and \( \Gamma(\theta_i) \) from (32) and (33).
6. Solve for \( \Omega \) at \( \theta_i \) from (35).
7. Compute the \( n \)-vector \( L_i \), defined as the left hand side of (36) (without the factor \( \rho^{-2} \)).
8. Store \( (\theta_i, L_i) \)

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9. Return to Step 1
10. Display $|L_i|$, for $i = 1, 2, \ldots, N$ for different norms and graphically determine $\min(|L_i|)$
11. Refine the grid size about a minimizing $\theta_i$

4 Numerical Difficulties

In step 3 of the previous section, $P_\omega(\theta)$ is computed as the solution of an $n \times n$ algebraic Riccati equation. For low values of $n$, the solutions found by a call to the MATLAB routine $lq_e$ (with arguments $A, \Sigma, \varphi_1$) were found to be quite accurate.

In step 6 of the previous section, $\Omega$ is computed as the solution of a Liapunov equation of order $n(n+2)$. Even for the case $n = 2$, large numerical errors were encountered using the MATLAB routine. Consequently, a block-wise (that is, partitioned) Liapunov equation solver was devised to take advantage of the sparseness of the particular form encountered. A chain of $2(n+2)$ matrix equations results, each of which is only of order $n \times n$.

Then the block-wise solution to (35) can be obtained by equating each of the following expressions to zero:

$$A(\theta_j)V_{i,j} + V_{i,j}A^T(\theta_j) + \Sigma \Sigma^T$$

$$A(\theta_0)V_{i,j} + V_{i,j}A^T(\theta_0) + \Sigma \Sigma^T$$

$$A(\theta_0)V_{i,j} + V_{i,j}A^T(\theta_0) + \Sigma \Sigma^T$$

$$A(\theta_j)V_{i,j} + V_{i,j}A^T(\theta_j) + (V_{i,j} - V_{j,i})e_j e_i^T$$

$$A(\theta_j)V_{i,j} + V_{i,j}A^T(\theta_j) + (V_{i,j} - V_{j,i})e_j e_i^T$$

$$A(\theta_j)V_{i,j} + V_{i,j}A^T(\theta_j) + (V_{i,j} - V_{j,i})e_j e_i^T$$

$$A(\theta_j)V_{i,j} + V_{i,j}A^T(\theta_j) + (V_{i,j} - V_{j,i})e_j e_i^T$$

where $j = 1, 2, \ldots, n$ in expressions (39) and (41).

In order to compute the vector $L_i$ which is the solution of the stationary point condition (36), computation can stop at this point, because the $j$th component of $L_i$ is $e_i^T V_{j,i}$. Thus, for the purpose of computing and storing $L_i$ (as in steps 7 and 8 of the previous section), only $2(n+2)$ matrix equations of order $n \times n$ need be solved (and step 6 need not be carried out completely). These equations are not Liapunov equations, except for the $V_{i,j}$ equations, because the solutions are in general non-symmetric.

For purposes of checking the solution of (35), $\Omega$ can be computed in its entirety by using the general formula for $k \leq j = 3, 4, \ldots, n + 2$

$$0 = A(\theta_j)V_{k,j} + V_{k,j}A^T(\theta_j) + (V_{k,j} - V_{j,k})e_j e_i^T$$

Solving for $\Omega$ in this manner gave much better solutions to (35) than the MATLAB Liapunov equation solution routine.

5 Results and Conclusions

Although the analysis of the previous sections applies to the estimation of pole parameters of linear systems of arbitrary dimension $n$, experiments were restricted to the two-dimensional case. As mentioned, even in the case $n = 2$ the partitioned solution of the Liapunov equation was required to accurately solve for the $8 \times 8$ matrix computed at each grid point. In the case $n = 2$, there are two pole parameters to estimate. The (true) system is in canonical form, that is, in the form

$$d\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sigma_1 & \sigma_2 \end{pmatrix} x dt + \begin{pmatrix} \theta_1^2 \\ \theta_2^2 \end{pmatrix} dw$$

where $\theta_1^2$ and $\theta_2^2$ are the true pole parameters.

As stated at the end of section 2, the convergence points are independent of $\rho$. Consequently, we chose to fix $\rho$ at 1, and to vary the remaining parameters $(\theta_1, \theta_2, \sigma_1, \sigma_2)$ in our experiments. Gridpoints were of the form $(\mu \theta_1^2, \mu \theta_2^2)$, where $\mu$ was a positive multiplicative scaling factor. In the illustrative plots below $\theta_0 = (\theta_1^2, \theta_2^2) = (-1, -2), \sigma = (1, 1), \mu = 2$ and $i, j = -7, \ldots, +7$. The semi-log placement of gridpoints in the spatial dimensions seemed natural in view of the desired stability relations $\theta_1^2 < 0, \theta_2^2 < 0$ and allowed the effective investigation of substantial portions of the plane.

Since $n = 2$ for our experimental cases, the possible convergence points are the zeroes of the two-vector field $L = (L_1, L_2)$. Figure 1 show a plot of the second component of $L$. Obviously, not a great deal of information can be gleaned from such plots alone, but inspection of the values (by printing out
values, rather than visually) does confirm that both components vanish at the center of the grid.

The next figure (Fig. 2) shows the negative of the Euclidean norm of \( L \), \(-\|L\| = -\sqrt{L_1^2 + L_2^2}\). This makes the minimum of \( \|L\| \) at the true value (grid center) visible as a maximum.

Fig. 3 shows the negative of the log likelihood function. This is the asymptotic limit of the output error covariance \( E\{Cz\mathcal{X}^TCT\} = e_1^TV_2,e_1 \) (where \( C = e_1^T \) is the output matrix). This function \( J_0(\bar{\theta}) \) in [3] can serve as a criterion function to determine the maximum likelihood values for \( \theta \). The fact that it has a maximum at the grid center corresponds to the fact that the true system is necessarily a maximum likelihood solution.

In none of our experiments did we find any indication of points of convergence other than those corresponding to the true system parameters. Although the possible existence of such additional points cannot be eliminated, since the search was not exhaustive, it appears very unlikely, given the nature of the cost surfaces obtained. We conjecture therefore, that there are no other points of convergence in the two-dimensional case. Note that the analogous result for dimension one has been obtained by a rigorous derivation ([4]).

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References


