A NEW ALGORITHM FOR PLOTTING ROOT LOCI (WITH APPLICATIONS)
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ABSTRACT
The position of a root on the s-plane depends on the characteristic equation and is a function of the 'root locus' gain. Using small perturbations and expanding in a Taylor's series a simple algorithm is obtained which permits calculation of the root locus without factoring the polynomial.

INTRODUCTION
System analysis using root loci requires construction of loci which are entirely in the complex plane. Problems involving slosh modes, mechanical resonances or the effects of feedback gains often involve loci beginning on complex poles and ending on complex zeros. When using special partitioning of the characteristic equation to apply the stability-equation method (1-4) essentially all poles and zeros are on the imaginary axis. Conventional techniques for longhand plotting of the root loci are not applicable to such problems, and conventional computer programs for calculating the loci using root-finding subroutines calculate of the roots (which is not necessary), usually giving insufficient resolution on the desired loci, and often are inaccurate if the polynomial is ill-conditioned.

It has been suggested (5) that the matrix form of Routh's criterion can be used. Another method applies the Newton-Raphson technique (6) and an expression for the root locus can be obtained by finding the gradients of two surfaces (7).

Since root loci start at a pole and end at a zero as the root locus gain is increased one can think of the motion of the root as a sort of velocity, or rate of change of position with respect to gain. The method developed in this paper is based on this concept of root motion: at any point on the root locus the incremental change root location caused by an incremental gain change can be computed—when added vectorially to the initial root point another point on the locus is determined.

DEVELOPMENT OF THE THEORY
The characteristic polynomial for any linear system can be represented as:

\[ F(s) = \sum_{i=0}^{n} a_i s^i = a_0 + a_1 s + a_2 s^2 + \ldots a_n s^n = 0 \]  

Such a polynomial can be manipulated into conventional root locus form by partitioning, which is the process of collecting the terms into two selected sets, and dividing through by one of the sets. Examples are

\[ (a_0 + a_1 s + a_2 s^2 + \ldots a_n s^n) \left(\frac{a_0 + a_1 s + a_2 s^2}{\ldots a_n s^n}\right) = 0 \]

and assuming n is an odd number

\[ (a_0 + a_2 s^2 + a_4 s^4) + (a_1 s + a_3 s^3 + \ldots a_n s^n - 1) = 0 \]

Equation 3b is commonly encountered in the stability curve method, and gives rise to imaginary poles and zeros.

Note that the general form of the polynomial is

\[ F(s) = F_N(s) + F_D(s) = 0 \]

where \( F_N \) and \( F_D \) are each polynomials, and the subscripts N and D are used to indicate "numerator" and "denominator" in the root locus form. \( K \) is a gain factor. Normal representation of equation 4 for use with root loci is

\[ \frac{F_N(s)}{F_D(s)} = -1 \]

Assume that the polynomial to be studied has been manipulated into the form of equation 4, and that \( F_N \) and \( F_D \) have been factored to determine the zeros and poles. Assume that we wish to start at a pole and calculate the segment of root locus from that pole. At the pole \( K = 0 \), so we increase this by \( \delta_k \) which causes the root to move from the pole an amount \( \delta_p \). Substituting in equation 4:

\[ \delta_k F_N(p + \delta_p) + F_D(p + \delta_p) = 0 \]

Assume that \( p = p_1 \neq z_1 \), and let \( p_1 \) be a simple pole, then the Taylor's Series expansion for equation 5 gives:

\[ F(s) = \sum_{i=0}^{n} a_i s^i = a_0 + a_1 s + a_2 s^2 + \ldots a_n s^n = 0 \]
Now \( \delta_F(p) = 0 \), and \( \epsilon_1(\delta_p) \) and \( \epsilon_2(\delta_p) \) are negligibly small, so equation 6 reduces to

\[
\delta_p F'_D(p) + \delta_K F_N(p) + \delta_p F'_N(p) = 0
\]

\[
\delta_p \left[ F'_D(p) + \delta_K F'_N(p) \right] = - \delta_K F_N(p)
\]

where \( F'_D(p) = \frac{dF_D(s)}{ds} \mid s = p \)

Equation 7 evaluates the first motion of the root away from a pole.

If we wish to start the calculations at some point on the root locus at which \( K \neq 0 \), (this includes calculating the second and subsequent points after leaving a pole), equation 5 becomes

\[
(K + \delta_K) F_N(p + \delta_p) + F_D(p + \delta_p) = 0
\]

The Taylor's Series expansion gives:

\[
(K + \delta_K) \left[ F_N(p) + \delta_p F'_N(p) + \epsilon_2(p) \right] + F_D(p) + \delta_p F'_D(p) + \epsilon_1(p) = 0
\]

This gives

\[
\delta_p = \frac{-\delta_K F_N(p)}{F_D(p) + (K + \delta_K) F_N(p)}
\]

Equations 7 and 10 are easily incorporated in a computer algorithm, as illustrated in Table 1. We chose to use a simulation language, DSL/360, as a matter of convenience.

**USE OF THE ALGORITHM**

The basic program can be used with any polynomial after a few preliminary manipulations. The polynomial must be partitioned into \( F_N(s) \) and \( F_D(s) \), from which \( F'_N(s) \) and \( F'_D(s) \) are then defined. A starting point must be chosen for the segment of locus that is to be calculated; this may be a pole if so desired, and \( F'_D(s) \) would have to be factored to find the pole; or it may be a zero, which requires factoring \( F_N(s) \); or it may be any known point on the root locus segment providing that the value of \( K \) is known in addition to the \( s \)-coordinates of the point.

Of course, \( s \) must be defined as a complex number. If the algorithm is started at a complex value of \( s \), the real and imaginary components are easily computed and the locus in the complex plane results. If the algorithm is started on the real axis it can compute points on the real axis up to the \( K \) value at which breakaway occurs, but cannot proceed into the complex plane because the incrementing process uses real numbers and cannot initiate a complex increment.

It should also be noted that \( \delta_K \) can be chosen either + or -, permitting the start point to be either a pole or a zero. The magnitude of \( \delta_K \) is relatively important, too large a value of \( \delta_K \) may introduce errors; fortunately the choice of \( \delta_K \) is not critical for most problems. If accuracy is essential \( \delta_K \) may be defined as a chosen fraction of \( K \).

**BASIC ILLUSTRATIONS**

Consider a third order unity feedback control system with forward transfer function:

\[
G(s) = \frac{K}{s(s + 1)(s + 5)}
\]

We shall calculate the conventional \( K \)-loci in the upper half of the \( s \)-plane. From \( G(s) \) it follows that:

\[
F_N(s) = 1, \quad F_D(s) = s^3 + 6s^2 + 5s, \quad F'_D(s) = 3s + 12s + 5.
\]

A starting point is found by using the Routh Criterion to find the stability limit: There is a root at \( s = j\sqrt{5}, K = 30 \). Using \( \delta_K = 0.1 \) and \( \delta_K = -0.1 \) the locus is calculated in both directions from the starting point. The root locus is shown on Fig. 1a, on which a comparison is made of root values calculated by this method, with values determined by a root-finding algorithm.

If a characteristic polynomial is:

\[
s^4 + Ks^3 + 109s^2 + 36Ks + 900 = 0
\]

We may wish to partition into even and odd power terms:

\[
1 + \frac{Ks^3 + 36Ks}{s^4 + 109s^2 + 900} = 0
\]

\[
1 + \frac{K(s^2 + 36)}{(s^2 + 9)(s^2 + 100)} = 0
\]

Obviously there are poles at \( s = \pm j3 \),
The root loci in the lower half s-plane are shown on Fig. 1b with comparison table for root values.

Use of feedback compensators also leads to root loci for which this algorithm is convenient. Consider the case of tachometer feedback \( k_1 s \) in an unstable unity feedback system for which the forward transfer function

\[ \frac{G(s)}{G(s)} = \frac{1}{s^4 + 30k_1 s + 900}, \]

is the characteristic equation

\[ s^4 + 30k_1 s + 900 = 0. \] (15)

A pole is at \( s = j5 \). The root locus for the upper half s-plane is shown on Fig. 2.

From these basic illustrations it is seen that the algorithm calculates both conventional and unconventional loci easily and with adequate accuracy.

**APPLICATION TO A FLEXIBLE MISSILE**

When the bending modes of a missile (or other structure) are included in the description of its dynamics, the characteristic equation of the system is often of high order. An example is given in equation 16:

\[
\begin{align*}
(s + 333)(s + 25)(s + 30) \cdot \\
(s^2 + 3s^2 + 605)(s^2 + 45.5s + 2660) \cdot \\
(s^2 + 2.51s + 3900)(s^2 + 3.99s + 22,980) \cdot \\
(s^2 + 42.2s + 2750)(s^2 + 13s + 3520) \cdot \\
(s^2 + 20.8s + 21,000) + K \cdot (333) \cdot (25) \\
(13.3) \cdot (-0.686) \cdot (2750) \cdot \\
(s + 2.26)(s + 53)(s - 53) \cdot \\
(s^2 - 152.2s + 14,500) \cdot \\
(s^2 + 153.8s + 14,500)(s^2 + 12s + 5810) \cdot \\
(s^2 + 22s + 12,800) = 0
\end{align*}
\]

This is the form of the equation after manipulation and factoring. Note that it is of 17th order. Using the method presented in this paper any segment of the root locus may be computed and studied without need for computation of any other segments. In a problem of this sort, several branches of the loci will cross the imaginary axis, and it is usually of interest to determine the value of frequency and gain at each of these crossings as a guide to design of stabilization. Thus one might choose to start computation at the pole located at \( s = -1.255 + j62.43 \), and compute the section of locus as illustrated on Fig. 3. Table 2 compares root values.

**STABILITY ANALYSIS OF NONLINEAR SYSTEMS**

For a feedback control system with cascaded nonlinearity the characteristic equation is conventionally written as

\[ 1 + N(A)G(s) = 0 \] (17)

where \( N(A) \) is the describing function of the nonlinearity. If the nonlinearity is single valued, several stability tests are available:

- a) Construct the ordinary root loci with \( N(A) \) as the gain parameter. If a limit cycle exists it is known to be at the intersection of the locus with the imaginary axis.

- b) Use frequency domain methods, applying the Nyquist criterion.

- c) Use the "stability curve" method\[1\].

- d) Use the method of this paper to calculate the stability curves.

Methods (a) and (b) are well known. The stability curve method requires that we partition the characteristic equation into even and odd power terms, and rearrange in root locus form, for example if

\[ 1 + N(A)G(s) = 1 + \frac{N(A)}{s(s+1)(s+5)} = 0 \] (18a)

\[ s^3 + 6s^2 + 5s + N(A) = 0 \] (18b)

from which

\[ 1 + \frac{6s^2 + N(A)}{s(s+5)} = 0 \] (18c)

Note that factors of both numerator and denominator give imaginary zeros and poles. From [1] a stability criterion is that the system is stable iff the poles and zeros alternate along the imaginary axis, i.e.,

\[-P_2 < C_1 < P_1 < Z_1 < C_2 < P_2 \] (19)

By inspection of equation 18c it is seen that the zeros are functions of \( N(A) \), and the stability limit is that value of \( N(A) \) for which \( Z_1 \) and \( Z_1 \) are the same as \( P_2 \) and \( P_2 \).
For more complex dynamics inspection is not adequate, so we can check stability by drawing a curve for the location of each zero and pole as functions of the parameter N(A). These are called the stability curves, and if they intersect the points of intersection define the stability limit. For the case of a single valued nonlinearity there is no significant advantage to the stability curve method, but if the nonlinearity is double valued it is often a convenient method.

Consider the system of Fig. 4, showing a second order system with backlash\(^*\). The characteristic equation is

\[
1 + \frac{N(A)w_n^2}{s^2 + 2\zeta w_n s} = 0
\]

Applying the original [1] stability equation method to equation 20

\[
s^2 + 2\zeta w_n s + \left[ g(A) + jb(A) \right] w_n^2 = 0 (21)
\]

let \( s = j\omega \), then

\[
-\omega^2 + j2\zeta w_n \omega + g(A)\omega_n^2 + jb(A)\omega_n^2 = 0
\]

The stability equations are

\[
-\omega^2 + g(A)\omega_n^2 = 0 (22)
\]

These give three curves (2 poles and one zero) plotting \( \omega \) as ordinate and \( A \) as abscissa, but we must factor the equations for each value of \( A \).

If we note that equation 20 can be put in the form

\[
F_D(s) + N(A)F_n(s) = 0
\]

this is the same form as equation 4. Repeating the manipulation applied to equation 4 we obtain

\[
\delta_P = -\frac{F_D(p) + \left[ N(A) + \delta \nu N'(A) \right] F_n(p)}{F_D(p) + \left[ N(A) + \delta \nu N'(A) \right] F_n(p)}
\]

Using \( \zeta = 0.094 \), \( \omega_n = 0.424 \) and \( B = 0.66 \), equation 24 is factored once to obtain starting values, then the stability curves can be calculated with equation 26, as shown on Fig. 5. The limit cycle occurs at the intersection of the curves.

**SYSTEMS WITH SEVERAL ADJUSTABLE PARAMETERS**

When systems have two adjustable parameters the motion of the roots as a function of these parameters can be determined using Siljak's methods [9], and sensitivity studies can be made by inspection of the curves. When a design method such as Horowitz's [10] is used, we wish to establish bounds on root motion as a function of parameter variation. For the case of two parameters \( a \) and \( b \), the characteristic equation may be of the form

\[
F_0(s) + aF_1(s) + bF_2(s) = 0 (27)
\]

From which we can derive

\[
\delta = -\frac{F_D(p) + (a + \delta_a)F_1(p) + (b + \delta_b)F_2(p)}{F_0(p) + (a + \delta_a)F_1(p) + (b + \delta_b)F_2(p)} (28)
\]

To illustrate the use, consider the system of Fig. 6, where

\[
G(s) = \frac{b_m}{s(s^2 + 2\zeta_0 w_p s + w_p^2)}
\]

\[
H(s) = \frac{s^2 + 2\zeta \omega_0 s + \omega_0^2}{\omega_0^2}
\]
The characteristic equation of the system is

$$s^3 + \left(2r_p \omega_p + \frac{k_m}{\omega_0^2}\right)s^2 + \left(\omega_p^2 + 2\zeta_0 \omega_0 \frac{k_m}{\omega_0^2}\right)s + k_m = 0 \tag{29}$$

Let $\zeta_p = 1.51/7.; \omega_p = 7.0; \ k_m = 192.$ $\zeta_0$ and $\omega_0$ are adjustable in the ranges:

- $0.4 \leq \zeta_0 \leq 0.2$
- $0.6 \leq \omega_0 \leq 0.3$

Define $a = \omega_0^2$ and $b = \zeta_0 \omega_0$, then the ranges of $a$ and $b$ are

- $0.36 \leq a \leq 0.09$
- $0.24 \leq b \leq 0.06$

and the characteristic equation becomes

$$192s^2 + a(s^3 + 3.02s^2 + 49s + 192) + b(34s) = 0 \tag{30}$$

Fig. 7 shows the range of parameter variation on the $a$ vs. $b$ plane and the boundary abcd is mapped onto the $s$-plane as also shown on Fig. 7. This was done using equation 28 and using $\delta_a = 0.1$ and $\delta_b = 0.1$.

The method is not restricted to two parameters. In the system of Fig. 6, assume that $k_m$ is also adjustable, in the range 150 $\leq k_m \leq 250.$ For three variables equation 28 becomes

$$\delta_p = -\frac{F_0(p) + (a + \delta_a)F_1(p) + (b + \delta_b)F_2(p) + \delta_{km}[F_{02}(p) + (a + \delta_a)F_{12}(p) + (b + \delta_b)F_{22}(p)]}{F_{01}(p) + (a + \delta_a)F_{11}(p) + (b + \delta_b)F_{21}(p)} \tag{31}$$

where $F_{01}, F_{11}, F_{21}$ and $F_{02}, F_{12}, F_{22}$ denote partial derivatives with respect to $s$ and $k_m$ respectively. Let $\delta_a = 0.1, \delta_b = 0.1, \delta_{km} = 20.$ Use of equation 31 maps the bounds in parameter space as shown on Fig. 8a, onto $s$-plane as shown on Fig. 8b.

CONCLUSIONS:

The root locus algorithm presented is a variation on the basic root locus method. It has some obvious computational advantages, and is particularly advantageous for detailed studies of selected segments of root loci.

The algorithm also is useful in studying nonlinear systems, and systems with several adjustable parameters. Unlike conventional root locus methods, it is capable of describing root motion as a function of the simultaneous variations in several parameters.

REFERENCES


(8) J.H. Blakelock, "Automatic Control of Aircraft and Missiles," John Wiley and Sons,


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**TABLE 1**

<table>
<thead>
<tr>
<th>K</th>
<th>COMPARISON TABLE - (K=0.1)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>PREVIOUS ALGORITHM ROOT</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.12 + 0.06j</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.12 + 0.16j</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.12 + 0.36j</td>
</tr>
</tbody>
</table>

**FIG 1a ROOT LOCUS FOR G(s) = K*(s-1) (s+3)**

**FIG 1b ROOT LOCUS FOR G(s) = K*(s+1) (s^2+3)**

**COMPARISON TABLE**

<table>
<thead>
<tr>
<th>K</th>
<th>SERIES ALGORITHM ROOT</th>
<th>ROOT FINDER VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>-0.12 + 0.06j</td>
<td>-0.11 + 0.06j</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.12 + 0.16j</td>
<td>-0.11 + 0.16j</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.12 + 0.36j</td>
<td>-0.11 + 0.36j</td>
</tr>
</tbody>
</table>

**FIG 2 ROOT LOCUS FOR G(s) = K*(s^2+3)**

**COMPARISON TABLE**

<table>
<thead>
<tr>
<th>K</th>
<th>SERIES ALGORITHM ROOT</th>
<th>ROOT FINDER VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>-0.12 + 0.06j</td>
<td>-0.11 + 0.06j</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.12 + 0.16j</td>
<td>-0.11 + 0.16j</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.12 + 0.36j</td>
<td>-0.11 + 0.36j</td>
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</tbody>
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TABLE 2. (WITH $\delta_k^2(z)$)

EXACT AND PREDICTED ROOTS OF THE 17th ORDER MISSILE SYSTEM OF FIG. 3.

<table>
<thead>
<tr>
<th>GAIN FACTOR ( K )</th>
<th>SERIES ALGORITHM ROOTS</th>
<th>EXACT CALCULATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>((-1.370+6.222j))</td>
<td>((-1.373+62.201j))</td>
</tr>
<tr>
<td>3.</td>
<td>((-1.047+6.053j))</td>
<td>((-1.630+61.515j))</td>
</tr>
<tr>
<td>5.</td>
<td>((-1.632+60.263j))</td>
<td>((-1.399+60.409j))</td>
</tr>
<tr>
<td>23.</td>
<td>(2.614+59.141j)</td>
<td>(2.611+59.141j)</td>
</tr>
<tr>
<td>35.</td>
<td>(3.956+59.278j)</td>
<td>(3.955+59.278j)</td>
</tr>
<tr>
<td>65.</td>
<td>(6.078+59.805j)</td>
<td>(6.077+59.805j)</td>
</tr>
<tr>
<td>119.</td>
<td>(8.300+60.689j)</td>
<td>(8.300+60.689j)</td>
</tr>
</tbody>
</table>

**Fig. 3** ROOT LOCI OF A FLEXIBLE MISSILE

**Fig. 4** A SECOND ORDER SYSTEM WITH BACKLASH

**Fig. 5** STABILITY CURVES FOR A SECOND ORDER SYSTEM WITH BACKLASH

**Fig. 6** CONTROL SYSTEM BLOCK DIAGRAM

**Fig. 7** CLOSED LOOP POLE REGION DETERMINED BY PARAMETER VARIATION REGION

**Fig. 8** CUBIC SPACE OF THE ADJUSTABLE PARAMETERS