STABILITY CONDITIONS FOR MULTIDIMENSIONAL FILTERS

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Summary

Recently developed[11]conditions for the stability of multidimensional filters are discussed. Unlike that in[11], the approach is inductive and intuitive. Apart from its intrinsic merit as an alternate method of reaching these conditions, this new approach is hoped to help simplify the tasks of numerically testing the stability conditions and of establishing methods for the stabilization of unstable multidimensional filtering operations.

I. Introduction

A fundamental frequency-domain stability criterion for multidimensional filters was established in[1] and[2]. According to this criterion, a spatially causal two-dimensional shift-invariant filter with transfer function:

\[ F(z_1, z_2) = \frac{B(z_1, z_2)}{A(z_1, z_2)} \]  

(1)

where \( A(z_1, z_2) \) and \( B(z_1, z_2) \) are polynomials, is stable if and only if:

\[ A(z_1, z_2) \neq 0 \text{ when } |z_1| < 1 \text{ and } |z_2| < 1 \]  

(2)

In the more general case of a \( k \)-dimensional spatially causal filter with a rational transfer function:

\[ F(z_1, ..., z_k) = \frac{B(z_1, ..., z_k)}{A(z_1, ..., z_k)} \]  

(3)

the corresponding stability criterion is:

\[ A(z_1, ..., z_k) \neq 0 \text{ when } |z_r| < 1, \ r = 1, ..., k \]  

(4)

Direct numerical implementation of (2) and (4) engenders extraordinary computational and storage requirements. An advantageous stability criterion, equivalent to (2) but computationally far simpler, was developed in [4]:

\[ A(0, z_2) \neq 0 \text{ when } |z_2| < 1 \]  

(5a)

\[ A(z_1, 0) \neq 0 \text{ when } |z_1| < 1 \]  

(5b)

Specific schemes for the numerical implementation of (5) were given in, among many other articles, [4] and [5]. In [6] it was shown that (5) can be generalized for application to filters of arbitrary dimension, and the following conditions were proven to be equivalent to (4):

\[ A(z_1,0,...,0) \neq 0 \text{ when } |z_1| < 1 \]  

(6)

\[ A(z_1, z_2,0,...,0) \neq 0 \text{ when } |z_1| = ... = |z_{k-1}| = 1 \]  

and \[ |z_k| < 1 \]

Both (5) and (6) have been highly useful as criteria for the stability of multidimensional filters, and have generated considerable research interest in specific methods for their numerical implementation.

The fact that (5) and (6) may be replaced by simpler and more flexible criteria was established in [11]. The purpose of this paper is to reexamine the criteria of [11] from the intuitively attractive Algebraic Function Theory point of view. This alternate approach clarifies the underlying ideas and, as such, may lead to progress in related important problems such as the problem of stabilization of multidimensional filters.

The precise mathematical foundation of the paper is established in section II. In section III a more flexible expression for (5) is given and a novel test is developed which is, at least conceptually, simpler than (5). The general stability test for multidimensional digital filters is given in the same section and is significantly simpler than (6), both conceptually and in terms of requisite computational effort. Numerical implementation of the stability tests is discussed in section IV.

II. Preliminary Development

Let \( A(z_1, z_2) \) in (1) represent a polynomial with real or complex coefficients of powers of \( z_1 \) and \( z_2 \). Let \( P(z_1) \) be the product of all polynomial factors of \( A(z_1, z_2) \) involving only \( z_1 \), and \( Q(z_2) \) the product of all factors involving only \( z_2 \). Then \( A(z_1, z_2) \) may be expressed as the following product:

\[ A(z_1, z_2) = P(z_1) A'(z_1, z_2) Q(z_2) \]  

(7)

Clearly, the zeroes of \( A(z_1, z_2) \) consist of all zeroes of \( P(z_1) \) and those isolated and all zeroes of \( A'(z_1, z_2) \) (which may be described by algebraic functions of \( z_1 \) and \( z_2 \)[7]). In this section we introduce the terminology used in the paper and quote or derive some properties of algebraic functions which are relevant to its purpose. The proofs of these properties are of little direct interest and are kept brief by referring whenever possible to literature in the theory of algebraic functions.

\[ A(z_1,0,...,0) \neq 0 \text{ when } |z_1| < 1 \]  

\[ A(z_1, z_2,0,...,0) \neq 0 \text{ when } |z_1| = ... = |z_{k-1}| = 1 \]  

and \[ |z_k| < 1 \]
The polynomial $A'(z_1, z_2)$ in (7) may be represented by:

$$A'(z_1, z_2) = a_0(z_1) z_2^n + a_1(z_1) z_2^{n-1} + \ldots + a_n(z_1)$$

where $a_i(z_1), i = 0, \ldots, n$ and $b_i(z_2), i = 0, \ldots, m$ are polynomial in $z_1$ and $z_2$ respectively. With no loss of generality, we assume that $a_i(z_1)$ and $b_i(z_2)$ are not identically zero. As is evident from (8), the roots of $A'(z_1, z_2) = 0$ trace the (multi-valued) algebraic function $[7]$ to $f(z_1)$, defined by

$$A'(z_1, z_2) = 0, \text{ and symmetrically, the algebraic function } z_2 = g(z_1) \text{, defined by } A'(g(z_1), z_2) = 0.$$ 

The value $z_1 = z_0^1$ is a singular point if $a_0(z_0^1) = 0$ or if $A'(z_0^1, z_2) = 0$, considered an equation in $z_2$, has multiple (finite or infinite) roots. An ordinary point of $z_2 = f(z_1)$ is one which is not singular. The following lemma provides conditions under which the number of such singular points is finite.

**Lemma 1**

Let $S_1$ and $S_2$ be sets of points of the $z_1$ plane and $z_2$ plane, respectively. If there exists no point $z_1 = c_1$ in $S_1$ such that $A(c_1, z_2) = 0$ for all $z_2$ and no point $z_2 = c_2$ in $S_2$ such that $A(z_1, c_2) = 0$ for all $z_1$, then:

(i) the factors $P(z_2)$ and $Q(z_2)$ in (7) have no zeroes in $S_1$ and $S_2$, respectively;

(ii) the algebraic functions $z_2 = f(z_1)$ and $z_1 = g(z_2)$ generated by the equation

$$A'(z_1, z_2) = 0$$

have at most a finite number of singular points.

**Proof**

Since $A(c_1, z_2) \neq 0$ if $c_1 \in S_1$ and $A(z_1, c_2) \neq 0$ if $c_2 \in S_2$, no zeroes of $P(z_2)$ are in $S_1$ and no zeroes of $Q(z_2)$ are in $S_2$. By its construction (7), $A'(z_1, z_2)$ contains no factors involving only $z_1$ or only $z_2$, which implies (7), section 11) that $f(z_1)$ and $g(z_2)$ have at most a finite number of singular points.

In the following, a curve $z(t), t_1 \leq t \leq t_2$ in a complex plane is called closed if $z(t_1) = z(t_2)$ and simply closed if $z(t_1) = z(t_2)$ but $z(t') \neq z(t'')$ for every other pair of distinct values $t'$, $t''$ on the interval $[t_1, t_2]$. A set $R$ of points of the complex plane is closed if it contains its boundary; it is connected if any two points in $R$ can be joined by a continuous curve lying entirely in $R$ and it is simply connected if further, the interior of $R$ and continuous on its boundary.

**Lemma 2**

Let $z_2 = f(z_1)$ be an algebraic function with a finite number of singular points.

(i) In any simply connected, closed subset $R$ of the $z_1$ plane, which is composed entirely of ordinary points of $z_2 = f(z_1)$, the values of $f(z_1)$ form a set $(f_1(z_1), i = 1, \ldots, n)$ of single-valued (ordinary) functions analytic in the interior of $R$ and continuous on its boundary.

(ii) Each function $z_2 = f_i(z_1), i = 1, \ldots, n$, maps any closed, connected curve in $R$ to a continuous, closed curve in the $z_2$ plane.

**Proof**

The assertion in part (i) is proven in [7] for the case where $R$ is open. Its extension to the case where $R$ is closed is straightforward since the number of singular points of $f(z_1)$ is finite and hence, any closed set composed entirely of ordinary points of $f(z_1)$ is a proper subset of an open set with the same property.

**Lemma 3**

Let $z_2 = g(z_1)$ be the algebraic function generated by $A'(z_1, z_2) = 0$, and assume that the number of singular points of $g(z_1)$ is finite. Then $z_2 = g(z_1)$ maps any closed curve in the $z_1$ plane to a continuous (although not necessarily closed) curves. A precise statement of this property is afforded by the following lemma.

Let $S_1$ and $S_2$ be closed, connected sets in the $z_1$ and $z_2$ planes respectively. Let $z_2 = f(z_1)$ be the algebraic function generated by $A'(z_1, z_2) = 0$ and assume that $f(z_1)$ has a finite number of singular points. If there exists a $a \in S_1$ such that $A'(a, z_2) \neq 0$ for all $z_2 \in S_2$, then there exists $a' \in S_1$, such that $a'$ is an ordinary point of $z_2 = f(z_1)$ and $A'(a', z_2) \neq 0$ for all $z_2 \in S_2$.

**Proof**

If $a$ is a singular point of $f(z_1)$, we may simply select $a' = a$. If $a$ is a singular point of $f(z_1)$, since the number of singular points is finite, there exists a sequence $a_0, a_1, \ldots, a_n$ of ordinary points of $f(z_1)$, converging to $a$. We contend that at least one $a_j$ of the elements of this sequence must be such that $A'(a_j, z_2) \neq 0$ for all $z_2 \in S_2$. In fact, if this is not the case, there must exist for each $j$ a complex number $z_2(j) \in S_2$ such that:

$$A'(a_j, z_2(j)) = 0$$

(9)
The sequence \( (z_2(j)) \) has at least one limit point \( z_2(m) \) which must be in \( S_2 \) since all \( z_2(j) \) are; also \( S_2 \) is closed. On taking limits of (9) and taking into account the continuity of \( A'(z_1,z_2) \), we find \( A'(a,z_2(m)) = 0 \). This obviously contradicts the stated assumptions and proves the above contention. Since \( a' = a_{10} \) is in \( S_1 \) and is an ordinary point of \( f(z_1) \), the lemma follows.

III. Stability Theorems

The key result of the paper is established in the following theorem. In the proof of the theorem and in the remainder of this section we shall use the following convenient notation:

\[
D_1 \triangleq \{ z_1 : |z_1| < 1 \}
D_2 \triangleq \{ z_2 : |z_2| < 1 \}
\]

Theorem 1

The following set of conditions is equivalent to (2), and thus necessary and sufficient for the stability of the spatially causal, digital filter (1).

(i) For some \( a, |a| \leq 1, A(a,z_2) \neq 0 \) when \( |z_2| < 1 \)

(ii) \( A(z_1,z_2) \neq 0 \) when \( |z_1| < 1 \) and \( |z_2| = 1 \)

Proof

Conditions (10) are obviously weaker than (2) and thus necessary for its validity. To prove sufficiency, assume that (10) are valid and note first that (10a) implies that there exists no value \( z_2 = c_2 \) in \( D_2 \) such that \( A(z_1,c_2) = 0 \) for all \( z_1 \). Likewise, (10b) implies that there exists no value \( z_1 = c_1 \) in \( D_1 \) such that \( A(c_1,z_2) = 0 \) for all \( z_2 \). By virtue of lemma 1 with \( S_1 = D_1 \) and \( S_2 = D_2 \), \( A(z_1,z_2) \) is zero for some \( z_1 \in D_1 \) and \( z_2 \in D_2 \) if and only if the polynomial \( A'(z_1,z_2) \) in (7) is zero in this region. It remains to show that the latter is impossible.

Note that the validity of (10a) obviously implies the validity of the following weaker condition:

For some \( a, |a| \leq 1, A'(a,z_2) \neq 0 \) when \( |z_2| < 1 \)

As shown by lemma 1, \( A'(z_1,z_2) = 0 \) generates an algebraic function \( z_2 = f(z_1) \) with at most a finite number of singular points. From lemma 4, applied with \( S_1 = D_1 \) and \( S_2 = D_2 \), it follows that there exists an ordinary point \( a' \) of \( z_2 = f(z_1) \) such that:

\[ |a'| \leq 1, A'(a',z_2) \neq 0 \] when \( |z_2| < 1 \)

Also, the validity of (10b) implies the validity of the following weaker condition:

\[ A'(z_1,z_2) \neq 0 \] when \( |z_1| < 1 \) and \( |z_2| = 1 \)

Thus, to establish sufficiency of (10) for the validity of (2), it suffices to prove that under (12) the polynomial \( A'(z_1,z_2) \) does not assume the value zero when
By definition of $f(z_1)$, the polynomial $A'(z_1,z_2)$ will have zeroes in the set $\{z_1 \in D_1', z_2 \in D_2\}$ if and only if one or more of the sets $E_i$ intersect $D_2$. The fundamental impetus of the theorem is the observation that this is possible only if either (a) one or more $E_i$ lie entirely in the interior of $D_2$, or, (b) one or more $E_i$ intersect the boundary $|z_2| = 1$ of $D_2$. The conditions (10) (or their surrogates (12)) are selected so as to eliminate each of these two possibilities. In fact, each $E_i$ is a closed and connected image of the whole set $D_1'$ and hence, if (a) above were true, then for each $z_1 \in D_1'$ (including $z_1 = a'$, since $a'$ is in $D_1'$) and some $z_2 \in D_2$ we would have $A'(z_1,z_2) = 0$. This is obviously impossible if (12a) holds. Likewise if (b) were true, then $A'(z_1,z_2) = 0$ for some $|z_1| \leq 1$, $|z_2| = 1$, which is impossible under (12b). Thus, (12) guarantees absence of zeroes of $A'(z_1,z_2)$ in the region $\{z_1 \in D_1', z_2 \in D_2\}$.

We conclude the proof of the theorem by following the arguments used in [4] and [12]. From what has already been said, if (12) is valid, all sets $E_i$ lie outside of $D_2$, and, hence, $|f_i(z_1)| > 1$ for all $z_1 \in D_1'$. In particular, for any one singular point $z_p$, we have:

$$|f(z_p + e^{i\theta})| > 1$$

whenever $0 < \theta$ and $e$ are such that $(z_p + e^{i\theta}) \in D_1'$. Clearly, $D_1'$ may be chosen so that $z_p$ is arbitrarily close to its boundary, and hence, in (14) is arbitrarily small. On taking limits of (14) as $\theta$ approaches zero and by virtue of the continuity of $f_i(z_1)$, we find $|f_i(z_p)| > 1$. By use of (12), this shows that under (12), $A'(z_1,z_2)$ is devoid of zeroes whenever $z_2 \in D_2$ and $z_1 \in D_1'$ (not merely when $z_1 \in D_1'$). The theorem follows.

Apart from its intrinsic interest as a generalization of (5), the criterion of theorem 1 leads to tangible simplification of known stability criteria. A simple criterion for two-dimensional filters is developed next.

**Theorem 2**

The following set of conditions is equivalent to (2) and, hence, necessary and sufficient for the stability of the spatially causal, digital filter (1):

(i) for some $a$, $|a| \leq 1$, $A(a_z, z_2) \neq 0$ when $|z_2| \leq 1$  \hfill (15a)

(ii) for some $b$, $|b| = 1$, $A(z_1, b) \neq 0$ when $|z_1| \leq 1$  \hfill (15b)

(iii) $A(z_1, z_2) \neq 0$ when $|z_1| = |z_2| = 1$  \hfill (15c)

In particular, with the choice $a = b = 1$, the above conditions become:

$$A(1, z_2) \neq 0 \text{ when } |z_2| \leq 1$$  \hfill (16a)

$$A(z_1, 1) \neq 0 \text{ when } |z_1| \leq 1$$  \hfill (16b)

$$A(z_1, z_2) \neq 0 \text{ when } |z_1| = |z_2| = 1$$  \hfill (16c)

**Proof**

Precisely as in the proof of theorem 1, (15a) and (15b) imply that $A(z_1, z_2) = 0$ for $z_1 \in D_1', z_2 \in D_2$ if and only if $A'(z_1, z_2) = 0$ in this region. As in theorem 1, the latter equation generates an algebraic function $z_1 = g(z_2)$ with m more than a finite number of singular points. Consider the images of the closed curve $|z_2| = 1$ induced on the $z_1$ plane by $z_1 = g(z_2)$:

$$C_i = \{z_1: z_1 = g_i(z_2), |z_2| = 1, i = 1, \ldots, m\}$$  \hfill (17)

From lemma 3 it follows that each $C_i$ is a continuous curve in the $z_1$ plane. Condition (10b) of theorem 1 will be violated only if either (a) at least one $C_i$ lies entirely in $D_1'$, or, (b) at least one $C_i$ intersects the boundary $|z_2| = 1$ of $D_1'$. Of the above, (a) is obviously impossible if (15b) is true and (b) is impossible if (15c) is true. Thus, (15b) and (15c) are sufficient for the validity of (10b). They are also necessary, because they are weaker than (10b). Clearly, (15b) and (15c) are equivalent to (10b) and in conjunction with (15a) (which is identical to (10a)), by virtue of theorem 1, necessary and sufficient for the validity of (2). The theorem follows.

It is emphasized that the theorem cannot be generalized by allowing $b$ to lie inside (instead of on) the unit circle. In particular, it could be tempting to speculate that (4) gives rise to the following conditions:

$$A(0, z_2) \neq 0 \text{ when } |z_2| \leq 1, A(z_1, 0) \neq 0 \text{ when } |z_1| \leq 1, A(z_1, z_2) \neq 0 \text{ when } |z_1| = |z_2| - 1$$

However, the latter conditions are not sufficient for the validity of (2). This is illustrated by the behavior of $A(z_1, z_2) = 0.5 - z_1 \cdot z_2$, which satisfies the above conditions, even though it is zero at $z_1 = 1, z_2 = 0.5$ and, hence, does not satisfy (2).

The main result of this section generalizes an appropriately chosen version of conditions (15) in theorem 2. It is proven by iterative application of these conditions.

**Theorem 3**

The following set of conditions is equivalent to (4) and thus necessary and sufficient for the stability of the spatially causal, digital filter (3):

(i) for some $b_1, \ldots, b_k$ such that $|b_r| = 1$, $r = 1, \ldots, k$, and for all $i = 1, \ldots, k, A(z_1, \ldots, z_k) \neq 0$ when $z_r = b_r$, $r \neq 1$ and $|z_1| \leq 1$  \hfill (18a)

(ii) $A(z_1, \ldots, z_k) \neq 0$ when $|z_1| = |z_2| = \ldots = |z_k| = 1$  \hfill (18b)

**Proof**

The necessity of (18) for the validity of (4) is obvious. We prove their sufficiency by induction on $k$. First, for $k = 2$ both necessity and sufficiency have been established in theorem 2. Assume sufficiency of (18) for the validity of (4), when $k = n$, and consider a polynomial $A(z_1, \ldots, z_{n+r})$ of $n + 1$ variables. Any such polynomial may always be regarded as a function of
the n variables \( z_2, \ldots, z_{n+1} \) with \( z_1 \) a parameter in range \( |z_1| \leq 1 \). By the inductive assumption of sufficiency of (18) when \( k = n \), we immediately find that 
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad z_1 = b_r, \ r \neq 1, \ r \neq i \quad \text{and} \quad |z_1| \leq 1, \ |z_i| \leq 1 \quad (19)
\]
and
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad |z_1| \leq 1, \quad |z_i| = \cdots = |z_{n+1}| = 1 \quad (20)
\]
Neither of the above is yet in the form (18) with \( k = n+1 \). Note, however, that (20) is weaker than the following condition:
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad |z_1| = |z_2| = \cdots = |z_{n+1}| = 1 \quad (21)
\]
which may be simplified by regarding now \( A(z_1, \ldots, z_{n+1}) \) as a function of the \( n \) variables \( z_1, \ldots, z_n \) with parameter \( z_{n+1} \) in the range \( |z_{n+1}| = 1 \). On applying once more (18) with \( k = n \), we find that (21) (and hence (20)) will be valid if:
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad |z_1| = |z_{n+1}| = 1 \quad (22)
\]
and, for each \( i, i = 1, \ldots, n \),
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad z_1 = b_r, \ r \neq n+1, \ r \neq i \quad \text{and} \quad |z_1| \leq 1, \ |z_{n+1}| \leq 1 \quad (23)
\]
where \( b_1 \) is an arbitrary complex number satisfying \( |b_1| = 1 \), and \( b_2, \ldots, b_n \) are chosen to be identical to the corresponding numbers in (19). Obviously, (23) is implied by the following condition:
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad |z_1| = 1, \ |z_i| \leq 1, \ |z_{n+1}| \leq 1 \quad (24)
\]
It has been shown that if (18) is sufficient for the validity of (4) when \( k = n \), then the set of conditions (19, 22, 24) is sufficient for the validity of (4) when \( k = n+1 \). It remains to show that (19), (22) and (24) are in turn implied by (18) with \( k = n+1 \), i.e., by the following set of conditions:
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad z_1 = b_r, \ r \neq i \quad \text{and} \quad |z_1| \leq 1 \quad (25)
\]
and
\[
A(z_1, z_2, z_4) \neq 0 \quad \text{when} \quad |z_1| = |z_2| = |z_4| = 1 \quad (26)
\]
Indeed, each of (19) is of the following form:
\[
A(z_1 b_2, \ldots, z_{n-1} b_{n-1}, z_n b_{n+1}, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad |z_1| \leq 1, \ |z_i| \leq 1.
\]
By virtue of theorem 2, the above is equivalent to:
\[
A(b_1 b_2, \ldots, b_{n-1} b_{n-1}, b_n b_{n+1} b_{n+1}, \ldots, b_{n+1}) \neq 0 \quad \text{when} \quad |z_1| \leq 1
\]
which may be simplified by regarding now \( A(z_1, \ldots, z_{n+1}) \) as a function of the \( n \) variables \( z_1, \ldots, z_n \) with parameter \( z_{n+1} \) in the range \( |z_{n+1}| = 1 \). On applying once more (18) with \( k = n \), we find that (21) (and hence (20)) will be valid if:
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad |z_1| = |z_{n+1}| = 1 \quad (22)
\]
and, for each \( i, i = 1, \ldots, n \),
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad z_1 = b_r, \ r \neq n+1, \ r \neq i \quad \text{and} \quad |z_1| \leq 1, \ |z_{n+1}| \leq 1 \quad (23)
\]
where \( b_1 \) is an arbitrary complex number satisfying \( |b_1| = 1 \), and \( b_2, \ldots, b_n \) are chosen to be identical to the corresponding numbers in (19). Obviously, (23) is implied by the following condition:
\[
A(z_1, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad |z_1| = 1, \ |z_i| \leq 1, \ |z_{n+1}| \leq 1 \quad (24)
\]
It has been shown that if (18) is sufficient for the validity of (4) when \( k = n \), then the set of conditions (19, 22, 24) is sufficient for the validity of (4) when \( k = n+1 \). It remains to show that (19), (22) and (24) are in turn implied by (18) with \( k = n+1 \), i.e., by the following set of conditions:
\[
A(z_1, z_2, z_4) \neq 0 \quad \text{when} \quad |z_1| = |z_2| = 1, \ |z_4| = 1 \quad (25)
\]
and
\[
A(z_1, z_2, z_4) \neq 0 \quad \text{when} \quad |z_1| = |z_2| = 1, \ |z_4| = 1 \quad (26)
\]
Indeed, each of (19) is of the following form:
\[
A(z_1 b_2, \ldots, z_{n-1} b_{n-1}, z_n b_{n+1}, \ldots, z_{n+1}) \neq 0 \quad \text{when} \quad |z_1| \leq 1, \ |z_i| \leq 1.
\]
By virtue of theorem 2, the above is equivalent to:
\[
A(b_1 b_2, \ldots, b_{n-1} b_{n-1}, b_n b_{n+1} b_{n+1}, \ldots, b_{n+1}) \neq 0 \quad \text{when} \quad |z_1| \leq 1
\]
\[ A(z_1, z_2, z_3, z_4, z_5) \neq 0 \text{ when } |z_1| = |z_2| = |z_3| = |z_4| = 1 \]  \hspace{1cm} (27e)

while (18) applied with \( b_1 = -1 \) and \( b_2 = b_3 = b_4 = b_5 = 1 \) gives \((k+1)\) conditions:

\[ A(z_1,1,1,1,1) \neq 0 \text{ when } |z_1| \leq 1 \]  \hspace{1cm} (28a)

\[ A(-1, z_2,1,1,1) \neq 0 \text{ when } |z_2| \leq 1 \]  \hspace{1cm} (28b)

\[ A(-1,1, z_3,1,1) \neq 0 \text{ when } |z_3| \leq 1 \]  \hspace{1cm} (28c)

\[ A(-1,1,1, z_4,1) \neq 0 \text{ when } |z_4| \leq 1 \]  \hspace{1cm} (28d)

\[ A(-1,1,1,1, z_5) \neq 0 \text{ when } |z_5| \leq 1 \]  \hspace{1cm} (28e)

and

\[ A(z_1, z_2, z_3, z_4, z_5) \neq 0 \text{ when } |z_1| = |z_2| = |z_3| = |z_4| = 1 \]  \hspace{1cm} (28f)

We first compare (27e) to (28f) assuming that (27e) is tested as in [6] and (28f) is tested by means of the procedures in [9]. Clearly, (28f) is conceptually simpler than (27e) and one may argue that this ensures commensurate computational simplicity under appropriate numerical implementation. In the absence of detailed numerical analysis of the techniques in [6] and [9], however, this cannot be substantiated and numerical superiority of either (27e) or (28f) cannot be claimed at the present time and with the present numerical implementation procedures.

Even if the above potential advantage of (28f) is overlooked, the computational advantage of (28) can be shown to be significant, if \( k \) is larger than 2, under any presently known numerical implementation methods. To demonstrate this, it will suffice to replace (28e) and (28f) by the stronger condition (27e).

The resulting test is

\[ A(z_1, 1, 1, 1, 1) \neq 0 \text{ when } |z_1| \leq 1 \]  \hspace{1cm} (29a)

\[ A(-1, z_2, 1, 1, 1) \neq 0 \text{ when } |z_2| \leq 1 \]  \hspace{1cm} (29b)

\[ A(-1, 1, z_3, 1, 1) \neq 0 \text{ when } |z_3| \leq 1 \]  \hspace{1cm} (29c)

\[ A(-1, 1, 1, z_4, 1) \neq 0 \text{ when } |z_4| \leq 1 \]  \hspace{1cm} (29d)

\[ A(z_1, z_2, z_3, z_4, 1) \neq 0 \text{ when } |z_5| \leq 1, \]  \hspace{1cm} \[ |z_1| = |z_2| = |z_3| = |z_4| = 1 \]  \hspace{1cm} (29e)

Condition (29e) is identical to (27e) and (29a), (27a) are both easily verified. However, the verification of (27b) to (27d) is arduous, while that of (29b) to (29d) may be effected by comparatively trivial application of the Schur-Cohn criteria. The resulting computational savings increase rapidly with \( k \) and could be far more important, from the computational point of view, than any advantage (28f) may hold over (27e).

References


