Performance Comparison of Different type of Reduced Order Modeling Methods

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Abstract—**Three techniques for constructing reduced order models are presented and compared. The techniques are based on: 1) Model Reduction Using the Routh Stability Criterion by V. Krishnamurthy and V. Seshadri.; 2) Higher Order Reduction by Coefficient Comparison by T. Manigandan and N. Devarajan 3) Clustering method for reducing order of linear system using Pade approximation by C.B. Vishwakarma and R. Prasad . In this paper, we have compared the reduced transfer functions obtained from all the three methods and thereafter computed the Relative Integral Square Error (RISE) for different numerical examples by using the above mentioned methods.**

Keywords—Routh Stability Criterion, Order Reduction, Transfer Function,Clustering method,Pade approximation

I. INTRODUCTION

Model reduction techniques are used in several situations: first, they are used when the actual order of the plant is unknown and empirical techniques yield a very high order model. Since the state variable feedback control design process produces a controller with the same order as the plant, it seems reasonable to restrict the order of the controller to be no more than the order of the plant model. So even when an accurate model is at hand, the economics of the situation might require a simpler controller. Higher order systems are difficult to use either for analysis or controller synthesis. Thus it motivates to reduce the order of a system without compromising the response of the system. Reduced order modeling is an important issue in control systems. In literature [1-15], various research papers have been published on reduced order modeling of higher order systems.

An algorithm which ensures that the reduced-order model derived by equating time-moments and by continued fraction synthesis will be stable, assuming the high-order system is stable[1]. It is observed that reduced order modeling using Routh criterion is a simple and effective method for reduced order modeling. The first paper on reduced order modeling using Routh criterion is appeared in 1975 by V. Krishnamurthy and V. Seshadri. This technique improves the drawbacks of Pade approximation method [3]. In this method, both the poles and zeros of the reduced model are considered to be free parameters and are obtained by minimizing the integral square error in impulse or step responses. When tested with a random input, the reduced models derived from impulse response deviations outperform those designed to minimize step response deviations [14].

A novel scheme to obtain a second order reduced model for stable linear time invariant continuous system and a PID controller is designed for the reduced second order model to meet the desired performance specifications by using pole-zero cancellation method by T. Manigandan, N. Devarajan and S. N. Sivanandam [15]. A mixed method for finding stable reduced order models of single-input-single-output large-scale systems using Pade approximation and the clustering technique. The denominator polynomial of the reduced order model is determined by forming the clusters of the poles of the original system, and the coefficients of numerator polynomial are obtained by using the Pade approximation technique. This method guarantees stability of the reduced order model when the original high order system is stable [4].

II. MODEL REDUCTION

Let us consider an n^{th} order transfer function

$$G_n = \frac{N_n(s)}{D_n(s)} = \frac{\sum_{i=0}^{n-1} a_i s^i}{\sum_{i=0}^{n} b_i s^i}$$

$$here\ a_i > 0; b_i > 0 \qquad (1)$$

The corresponding stable k^{th} $(k<n)$ order reduced model is of the form

$$R_k = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{k-1} d_i s^i}{\sum_{i=0}^{k} e_i s^i}$$

$$here\ d_i > 0; e_i > 0 \qquad (2)$$

where $d_i(0 \le i \le k-1)\ and\ e_i(0 \le i \le k)$ are scalar constants.

In this paper, assuming the original system described by equation (1), the problem is to find a reduced order model in the form of equation (2) such that the reduced order

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model retains the important characteristics of the original system approximates its response as closely as possible for the same type of inputs.

III. TECHNIQUES OF REDUCING MODEL

A. Method 1:(Model Reduction Using the Routh Stability Criterion by V. Krishnamurthy and V. Seshadri) [3]

Let the transfer function of the high-order system be

$$G(s) = \frac{b_{11}s^m + b_{21}s^{m-1} + b_{12}s^{m-2} + \cdots}{a_{11}s^n + a_{21}s^{n-1} + a_{21}s^{n-2} + \cdots} \qquad (3)$$

Where $m<n$.

The Routh Stability arrays for the numerator and denominator polynomials of (3) are given below in Tables I and II, respectively. The first two rows of Table I and II are directly constructed from the coefficients of these two polynomials, respectively.

TABLE I

Numerator Stability Array

b_{11}	b_{12}	b_{13}	b_{14}	.	.	.	.
b_{21}	b_{22}	b_{23}	b_{24}	.	.	.	.
b_{31}	b_{32}	b_{33}	.	.	.	.	.
b_{41}	b_{42}	b_{43}	.	.	.	.	.
.	.	.	.				
.	.	.	.				
$b_{m,1}$							
$b_{m+1,1}$							

TABLE II

Denominator Stability Array

a_{11}	a_{12}	a_{13}	a_{14}	.	.	.	.
a_{21}	a_{22}	a_{23}	a_{24}	.	.	.	.
a_{31}	a_{32}	a_{33}	.	.	.	.	.
a_{41}	a_{42}	a_{43}	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.		
$a_{n-2,1}$	$a_{n-2,2}$						
$a_{n-1,1}$	$a_{n-1,2}$						
$a_{n,1}$							
$a_{n+1,1}$							

Generalizing this, the transfer function of a system with reduced order k ($\leq n$) can easily be constructed with *(m+2-k)th* and *(m+3-k)th* rows of Table I and *(n+1-k)th* and *(n+2-k)th* rows of Table II as in (4).

$$R_k(s) = \frac{b_{(m+2-k),1}S^{K-1} + b_{(m+3-k),1}S^{K-2} + \cdots}{a_{(n+1-k),1}S^K + a_{(n+2-k),1}S^{K-1} + \cdots} \qquad (4)$$

For $k>(m+1)$, the first two rows of table I should be used for the numerator polynomial, while for $k=1$, only the last row should be used.

B. Method 2:(Higher Order Reduction by Coefficient Comparison by T. Manigandan and N. Devarajan) [15]

The procedure for determining the reduced order model is briefly described below:

The n^{th} order original system given in equation (1) is equated to the k^{th} order reduced model with unknown parameters represented by equation (2).

Hence,

$$G_n(s) = R_k(s)$$

$$\frac{a_0 + a_1s + a_2s^2 + \ \ldots. + a_{n-1}s^{n-1}}{b_0 + b_1s + b_2s^2 + \ \ldots. + a_ns^n} = \frac{d_0 + d_1s + d_2s^2 + \ \ldots. + a_{k-1}s^{k-1}}{e_0 + e_1s + e_2s^2 + \ \ldots. + e_ns^k} \qquad (5)$$

On cross-multiplying and rearranging the equation (5)

$$a_0e_0 + (a_0e_1 + a_1e_0)s + (a_0e_2 + a_1e_1 + a_2e_0)s^2 + a_{n-1}e_ks^{n-1+k} = b_0d_0 + (b_0d_1 + b_1d_0)s + (b_0d_2 + b_1d_1 + b_2d_0)s^2 + b_nd_{k-1}s^{n-1+k} \qquad (6)$$

Equating the coefficients of the corresponding terms in the equation (6), the following relations are obtained:

$$a_0e_0 = b_0d_0$$

$$a_0e_1 + a_1e_0 = b_0d_1 + b_1d_0$$

$$a_0e_2 + a_1e_1 + a_2e_0 = b_0d_2 + b_1d_1 + b_2d_0$$

. . .

. . .

. . .

$$a_0e_{k-1} + a_1e_{k-2} + a_2e_{k-3} + \ \ldots\ldots = b_0d_{k-1} + b_1d_{k-2} + b_2d_{k-3} + \ \ldots\ldots$$

$$a_0e_k + a_1e_{k-1} + a_2e_{k-2} + \ldots\ldots = b_1d_{k-1} + b_2d_{k-2} + b_3d_{k-3} + \ldots\ldots$$

$$a_1e_k + a_2e_{k-1} + a_3e_{k-2} + \ldots\ldots = b_1d_{k-1} + b_3d_{k-2} + b_4d_{k-3} + \ldots\ldots$$

. . .

. . .

. . .

$$a_{n-1}e_k = b_nd_{k-1}$$

The unknown parameters are determined by taking any positive values for d_0 or e_0, for simplification, choosing $d_0 = 1$ or $e_0 = 1$, and using the above relations, the unknown parameters are evaluated. The reduced order models are tested for stability using Routh array.

C. *Method 3: (Clustering method for reducing order of linear system using Pade approximation by C.B. Vishwakarma and R. Prasad) [4]*

This method consists of the following two steps:

Step-1: *Determination of the denominator of k^{th} order reduced model, using the clustering technique [14].*

The criterion for grouping the poles in one particular clustering is based on relative distance between the poles and the desired order in the process of reduced order modeling.

Since each cluster may be finally replaced by a single (pair of) real (complex) pole, the following rules are used for clustering the poles to get the denominator polynomial for reduced order models.

i. Separate clusters should be made for real poles.
ii. Poles on the jw-axis have to be retained in the reduced order model.

The cluster center can be formed using a simple method known as 'Inverse Distance Measure', which is explained as follows:

Let, r real poles in one clustering be ($p_1, p_2, p_3, \ldots\ldots\ldots P_r$) then the 'Inverse Distance Measure' (IDM) criterion identifies the cluster center as

$$p_c = \left\{\left(\sum_{i=1}^{r}\left(\frac{1}{p_i}\right)\right) \div r\right\}^{-1} \quad (7)$$

Where p_c is cluster center from r real poles of the original system.
Let m pair of complex conjugate poles in a cluster be $[(\alpha_1 \pm j\beta_1) + (\alpha_2 \pm j\beta_2), \ldots\ldots\ldots\ldots\ldots (\alpha_m \pm j\beta_m)]$ then the IDM criterion identifies the complex cluster center in the form of $A_c \pm jB_c$.

$$A_c = \left\{\left(\sum_{i=1}^{m}\left(\frac{1}{p_i}\right)\right) \div r\right\}^{-1} \text{ and } B_c = \left\{\left(\sum_{i=1}^{m}\left(\frac{1}{B_i}\right)\right) \div r\right\}^{-1} \quad (8)$$

For getting the denominator polynomial for k^{th} order reduced model, one of the following cases may occur.

Case 1: If all cluster centers are real, then denominator polynomial for k^{th} order reduced model can be obtained as

$$D_k(s) = (s-p_{c1})(s-p_{c2})\ldots\ldots\ldots\ldots(s-p_{ck}) \quad (9)$$

Where $p_{c1}, p_{c2,\ldots\ldots\ldots\ldots} p_{ck}$ are $1^{st}, 2^{nd}, \ldots\ldots\ldots, k^{th}$ cluster center.

Case 2: If (k-2) cluster centers are real and one pair of cluster center is complex conjugate, then $D_k(s)$ can be obtained as

$$D_k(s) = (s-p_{c1})(s-p_{c2})\ldots\ldots\ldots(s-p_{c(k-2)})(s-p_{c1})(s-p_{c1}) \quad (10)$$

Where p_{c1} and p_{c1} are complex conjugate cluster centers or $p_{c1}=A_c+jB_c$ and $p_{c1}=A_c-jB_c$.

Case 3: If all the cluster centers are complex conjugate, then reduced denominator polynomial can be taken as

$$D_k(s) = (s-p_{c1})(s-p_{c1})(s-p_{c2})(s-p_{c2})\ldots\ldots\ldots(s-p_{ck/2})(s-p_{ck/2}) \quad (11)$$

Step-2: *Determination of the numerator of k^{th} order reduced model, using Pade approximation* [15].

The original n^{th} order system can be expanded in power series about s = 0 as

$$G(s) = \frac{\sum_{i=2}^{n-1} e_i s^i}{\sum_i^n f_i s^i} = c_0 + c_1 s + c_2 s^2 + \ldots\ldots\ldots \quad (12)$$

The coefficients of the power series expansion are calculated as follows:

$c_0 = e_0$

$$c_i = \frac{1}{f_i}\left[e_i - \sum_{j=1}^{i} f_i c_{i-j}\right], i > 0 \quad (13)$$

$e_i = 0,\ i > n-1$
The reduced k^{th} order model is written as

$$G(s) = \frac{N_k(s)}{D_k(s)} = \frac{\sum_{i=0}^{k-1} a_i s^i}{\sum_{i=0}^{k} b_i s^i} \quad (14)$$

Here $D_k(s)$ is known through equations (9-11).

For $G_k(s)$ of equation (14) to be Pade approximates of G(s) of equation (12), we have

$a_0 = b_0 c_0$

$a_1 = b_0 c_1 + b_1 c_0$

…………
…………

$a_{k-1} = b_0 c_{k-1} + b_1 c_{k-2} + \ldots\ldots\ldots\ldots + b_{k-1} c_1 + b_k c_0 \quad (15)$

the a_j;$j=0.1,2\ldots\ldots\ldots\ldots k-1$ can be found by solving the above *k* linear equations.

IV. Numerical Examples

Reducing fourth order system using all the three Methods :

Consider the fourth order transfer function

$$G_4(s) = \frac{2400 + 1800s + 496s^2 + 28s^3}{240 + 360s + 204s^2 + 36s^3 + 2s^4}$$

Using method 1 :

The completed numerator and denominator tables are as shown below:

Numerator Table	
28	*1800*
496	*2400*
1664.516	*0*
2400	

Denominator Table		
2	*204*	*240*
36	*360*	
184	*240*	
313.043	*0*	
240		

The reduced-order transfer functions can immediately be derived from (4) for any value of *k(≤n)*. For *k=2* and *5*,

$$R_2(s) = \frac{2400 + 1662.516s}{240 + 313.043s + 184s^2}$$

Using method 2 :

Consider a second order reduced model represented by

$$R_k(s) = \frac{d_0 + d_1 s}{e_0 + e_1 s + e_2 s^2}$$

Where d_0, d_1, e_0, e_1 and e_3 are unknown parameters.d_0=1 or e_0=1, and the unknown parameters are evaluated using the method 2.

The second order reduced model is obtained as

$$R_2(s) = \frac{410.256 + 14s}{14.0256 + 29.5897s + s^2}$$

Using method 3:

The transfer function G(s) can be expanded about s = 0 as

$$G(s) = 1 - 0.75s + .5817s^2 - 0.37335s^3 + \ldots$$

Using step-1, two cluster centers from the real poles -1,-2 and -3,-4 can be formed from equation (3) as

$$P_{c1} = \left[\left(\frac{1}{-1.35} + \frac{1}{-0.6934}\right) \div 2\right]^{-1} = -0.9162$$

$$P_{c2} = \left[\left(\frac{1}{-3} + \frac{1}{-4}\right) \div 2\right]^{-1} = -3.42857$$

Therefore, denominator $D_2(s)$ can be synthesized using equation (5) and is given by

$$D_2(s) = (s + 2.0751 \pm j0.9162)$$
$$= 5.1454 + 4.1502s + s^2$$

The reduced second order model can be taken in the form of

$$G_2(s) = \frac{a_0 + a_1 s}{5.1454 + 4.1502s + s^2}$$

Now using step-2

$$a_0 = kb_0 c_{0=} 51.454$$

$$a_1 = b_0 c_1 + b_1 c_0 = 20.363$$

Therefore, finally the 2nd order reduced model is taken as

$$G_2(s) = \frac{51.454 + 20.363s}{5.1454 + 4.1502s + s^2}$$

This reduced model is stable and non-minimum phase.

V. Method Of Comparison

In order to compare the accuracy of both methods, the relative integral square error RISE between the transient parts of the reduced models and the original system is calculated using Matlab/Simulink.

The relative integral square error RISE is defined as

$\text{RISE}=\int_0^{\infty}[y(t)-y_r(t)]^2 \,/\, \int_0^{\infty}[y(t)-y(\infty)]^2\,dt \ldots\ldots()$

Where y(t) and $y_r(t)$ are step responses of the original and reduced model respectively and y(∞) is the steady-state value of step response of the original system.

VI. Comparative Results

A. *Comparison of Different Parameters:*

Table 2:4TH Order Model Reduction

Method of Reduction	Reduced Model	RISE
Method-1	$\dfrac{2400+1662.516s}{240+313.043s+184s^2}$	0.0304
Method-2	$\dfrac{401.21+35s}{36.63+1.436s+s^2}$	0.0110
Method-3	$\dfrac{51.454+20.363s}{5.1454+4.1502s+s^2}$	0.1472

B. *Comparison of step responses in time domain:*

In fig. 1 the step responses of 4th order original system is compared with the step responses of reduced order model generated by three different methods in time domain.

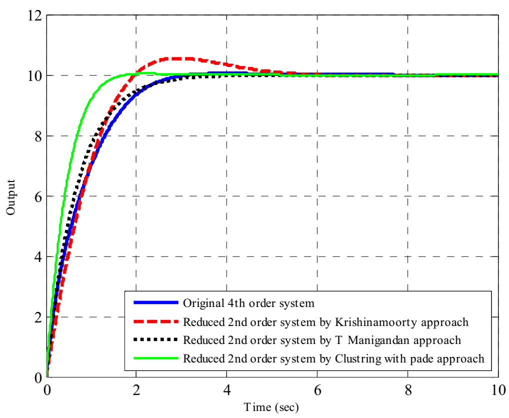

Fig 1: Comparison of step responses (4^{th} order system)

C. *Comparison of step responses in frequency domain:*

In fig. 2 step responses of 4^{th} order original system is compared with the step responses of reduced order models generated by three different methods in frequency domain.

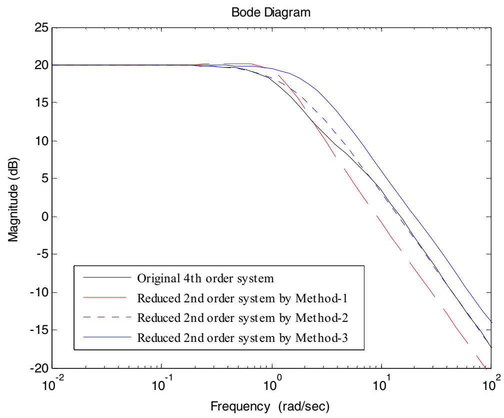

Fig 2: Comparison of step responses (4^{th} order system)

The graph clearly shows that the step responses of the three reduced order models in both time and frequency domain closely matches with that of original system.

VII. Conclusion

In this paper three reduced order models of a fourth order system has been obtained by applying three different techniques as suggested by V. Krishnamurthy and V. Seshadri, T. Manigandan and N. Devarajan, and C.B. Vishwakarma and R. Prasad respectively. Here we have compared the transfer functions of reduced order models obtained from all the three methods in terms of Relative Integral Square Error (RISE) for fourth order system. In addition to this the step response of 4th order original system is compared with the step responses of reduced order models generated by three different methods in both time and frequency domain. The estimated value of RISE by second method is lowest and hence this method is best among other methods. All the three methods give very good approximation of higher order systems. The step responses of the three reduced order models in both time and frequency domain closely matches with that of original system. On the basis of transient and steady state parameters (both time domain and frequency domain) applied on higher order systems, Method II gives comparatively better results.

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