Changing a rotation matrix $A$ to $-A$ interchanges proper and improper, changes the sign of the rotation angle, but does not change the axis. From this we obtain

$$-I + \frac{\sin \theta}{\lambda} L - \frac{1 + \cos \theta}{\lambda^2} L^2, \quad L = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

which is the matrix of a rotatory reflection about an axis in the direction $[a, b, c]$ through the angle $\theta$.

**Products of reflections**

The product of the two reflections in planes perpendicular to $\ell$ and $m$ is a proper rotation with axis in the direction perpendicular to both $\ell$ and $m$ and through an angle twice that between $\ell$ and $m$. This is an easy geometric fact. It can also be argued using matrices for the reflections as $A = -(I + 2(L^2/\lambda^2))$ and $B = -(I + 2(M^2/\mu^2))$, recalling that the angle of rotation is found by $1 + 2\cos \theta = \text{tr}(AB)$, and computing $\text{tr}(L^2M^2)$ from $(L^2 + \lambda^2I)(M^2 + \mu^2I) = \ell^T m \ell m$. Conversely, every proper rotation is the product of two reflections; we simply choose $\ell$ and $m$ perpendicular to the axis and separated by a suitable angle.

From the argument in the last section, we know that an improper rotation is the product of a reflection and a proper rotation, so three reflections suffice to obtain improper rotations.

**Theorem**: Every rotation of space (proper or improper) is the product of no more than three reflections.

This is a special case of a theorem going back to E. Cartan. A general formulation can be found in the beautiful but difficult book by Artin. An elementary discussion of direct and opposite isometries of space that includes translations and their kin appears in Coxeter. The utility of this particular theorem for computer graphics is not the statement but the connection with linear algebra, as discussed here.

**References**


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